# NATIONAL OPEN UNIVERSITY OF NIGERIA

## FACULTY OF SCIENCE

## DEPARTMENT OF COMPUTER SCIENCE

**COURSE CODE: CIT 425**

**COURSE TITLE: Operations Research**

**Course Code**          **CIT 425**

**Course Title**          **Operations Research**

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# MODULE ONE   CONCEPTS OF OPERATIONS RESEARCH

UNIT 1:      Development of Operations Research

UNIT 2:      Modelling in Operations Research

# MODULE TWO  MATHEMATICAL PROGRAMMING

UNIT 1:      Optimization problem

UNIT 2:      Linear programming (LP)

UNIT 3:      Linear Programming (Simplex Method)

UNIT 4:      INTEGER PROGRAMMING

# MODULE THREE  MULTI STAGE DECISION PROCESSES

**UNIT 1:      Dynamic Programming**

# MODULE FOUR  SCHEDULING MODELS

UNIT 1:      TRANSPORTATION MODEL

UNIT 2:      Assignment Model

# MODULE ONE   CONCEPTS OF OPERATIONS RESEARCH

## UNIT 1 DEVELOPMENT OF OPERATIONS RESEARCH (O.R)

**CONTENTS**

## 1.0   INTRODUCTION

Operations Research has existed since the beginning of recorded history. As far back as World War II, operations research techniques have been developed to assist the Military during the war. Today, many organizations employ a staff of operation researcher or management science personnel or consultants to apply the principles of operations research to management problems.

There are actually several approaches in interpreting, analyzing and solving business problems; viz:

  i. The Conventional Approach: This involves following past techniques and solutions to solving present problems. This method if state; it offers little or nothing to the advancement in management.
 ii. The Observation Approach: A method of watching and learning from other managers in similar situation.
iii. Systematic/Scientific Approach: Utilizes concept of theoretical systems, which may be somewhat different from the actual problem under study. This is the operations research method of solving management problem.

The successful use of operations research techniques usually results in a solution that is timely, accurate, flexible, economical, reliable and easy to understand and use. We will be familiar with the limitations, assumption and specific applicability of the techniques.

## 2.0    OBJECTIVES

By the end of this unit the reader will be able to:

- ✓ Understand the meaning of Operations Research

- ✓ Highlight the historical development of Operations Research

- ✓ Describe the scientific nature of Operations Research

- ✓ Identify the importance and uses of Operations Research with

   respect to the various topics to be treated in the study

- ✓ State the limitations of Operations Research

## 3.0    MAIN TEXT

## 3.1    <u>Definition</u>

Defining Operations Research itself is very difficult. Like many other subjects that developed pragmatically and shade imperceptibly into adjoining subjects, it is more easily recognized than defined.

Generally speaking, operations research is an approach to the analysis of operations that to a greater or lesser extent adopts:

i.   Scientific method (observation, hypothesis, deduction and experimentation as far as possible).
ii.  The explicit formulation of complex relationships.
iii. An inter-disciplinary nature.
iv.  A non-partisan attitude.

Operational Research can also be regarded as a scientific approach to the analysis and solution of management problem.

The council of the United Kingdom Operational Research Society defines Operational Research as "the attack of modern science on complex problems, arising in the direction and management of large systems of men, machines, materials and money in industry, business, government and defence. It goes on to state the distinctive approach as to develop a scientific model of the system; incorporating measurement of factors such as chance and risk, in order to predict and compare the outcomes of alternative decisions, strategies and controls. The purpose is to help management to determine its policy and action scientifically".

It is also worth pointing out that an Operations Research project is often a team effort that involved people drawn from many different backgrounds including: Accountants, Engineers, Mathematicians, Statisticians and Scientist as well as the operations research experts themselves.

## 3.2 Historical Background of O.R

Pre-World war II: The roots of OR are as old as science and society. Though

the roots of OR extend to even early 1800s, it was in 1885 when Ferderick W.

Taylor emphasized the application of scientific analysis to methods of production, that the real start took place.

Another man of early scientific management era was Henry L. Gantt. Most job-scheduling methods at that time were rather haphazard. A job, for instance, may be processed on a machine without trouble but then wait for days for acceptance by the next machine. Gantt mapped each job from machine to machine, minimizing every delay. Now with the Gantt procedure it is possible to plan machine loadings months in advance and still quote delivery dates accurately.

In 1917, A.K.Erlang, a Danish mathematician, published his work on the problem of congestion of telephone traffic. The difficulty was that during busy periods, telephone operators were many, resulting in delayed calls. A few years after its appearance, his work was accepted by the British Post Office as the basis for calculating circuit facilities.

The well known economic order quantity model is attributed to F.W. Harris, who published his work on the area of inventory control in 1915.

During the 1930s, H.C. Levinson, an American astronomer, applied scientific analysis to the problems of merchandising. His work included scientific study of customers' buying habits, response to advertising and relation of environment to the type of article sold.

However, it was the First Industrial Revolution which contributed mainly towards the development of OR. Before this revolution, most of the industries were small scale, employing only a handful of men.

The advent of machine tools-the replacement of man by machine as a source of power and improved means of transportation and communication resulted in fast flourishing industry. It became increasingly difficult for a single man to perform all the managerial functions ( of planning, sale, purchase, production, etc.). Consequently, a division of management function took place. Managers of production, marketing, finance, personnel, research and development etc., began to appear. With further industrial growth, further subdivisions of management functions took place. For example ,production department was sub-divided into sections like maintenance, quality control, procurement, production planning, etc.

World War II:  During World War II, the military management in England called on a team of scientists to study the strategic and tactical problems of air and land defence. This team was under the direction of Professor P.M.S. Blackett of University of Manchester and a former naval officer. "Blackett circus", as the group was called, included three physiologist, two mathematical physicists, one astrophysicist, one army officer, one surveyor, one general physicist and two mathematicians. Many of these problems were of the executive type. The objective was to find out the most effective allocation of limited military resources to the military operations and to the activities within each operation. The application included the effective use of newly invented radar, allocation of British Air Force Planes to missions and the determination of best patterns for searching submarines. This group of scientists formed the first OR team.

The name operations research ( or operational research) was apparently coined because the team was carrying out research on (military)operation.the encouraging results of these effort led to the information of more such teams in British armed services and the use of scientific teams soon spread to western allies-the united states, Canada and France. thus through this scince of operation research originated in England,the united states soon took the lead.in united state these OR teams helped in developing stategiesfrom mining operations, inventing new flight patterns and planning of sea mines.

(b) Post-world war II : immediately after the war,the success of military teams attracted the attention of industrial managers who were seeking solutions to their problems. Industrial operation research in U.K. and U.S.A. developed along different lines. In U.K., the critical economic situation required drastic increase in production efficiency and creation of new markets.nationalisation of a few key industries further increased the potential field for OR. Consequently OR soon spread from military to government, industrial, social and economic planning.

In U.S.A.the situation was different. Impressed by its dramatic success in U.K., defence operations research in U.S.A was increased. Most of the war experienced OR workers remained in military service. Industrial executives did not call for much help because they were returning to the peace-time situation and many of them believed that it was merely a new application of an old technique. Operation research by a variety of names in that country such as operational analysis, operation evaluation, systems analysis, system evaluation, system research and management science.

The progress of industrial operational research in U.S.A. was due to advent of second industrial revolution which resulted in *automation*-the replacement of man by machine as a *source of control*.the new revolution began around 1940s when electronic computers became commercially available. The electronic brains processed tremendous computational speed and information storage. But for these digital computers, operation research with its complex computational problems could not have achieved its promising place in all kinds of operational environments.

In 1950, OR was introduced as a subject for academic study in American universities since then this subject has been gaining ever increasing importance for the students of Mathematics, Statistics, Commerce, Economics, Management and Engineering. To increase the impact of operation research, the Operations Research Society of America was formed in 1950. In 1953, the Institute of Management Sciences (IMS) was established. Other countries followed suit and in 1959 International Federation of OR began to appear. Some of them

( in English ) are:

*Operations Research

*Opsearch

*Operational Research Quarterly

*Management Science

*Transportation Science

* Mathematics of Operations Research

* International Journal of Game Theory, etc.

Today, the impact of operations research can be felt in many areas. This is shown by the ever increasing number of educational institutions offering this subject at degree level. Of late, OR activities have spread to diverse fields such as hospitals, libraries, Planning, transportation systems, management, defence, etc.

### 3.3    Basic facts about O.R as a concept

The following are basic facts about Operations Research:

- It is a science-based approach to analyzing problems and decision situations to aid solving such problems and decision-making. It is therefore a practical activity, although based on the theoretical construction and analysis.
- It is an approach and an aid to problem-solving and decision-making.
- Its distinctive approach is facts-finding and modeling.
- It examines functional relations (i.e. functions of a system and their related components) from a system overview.
- It utilizes interdisciplinary mixed-team approach to solving management problems.
- It adopts the planned approach (updated scientific method which reflects technological advancement as the computer) to management problems.
- It helps to discover new problems as one problem is being solved.

### 3.4.    O.R. as an adaptation of the Scientific Approach

Operations Research is most fundamentally science-based.

It is so by adoption and adapting the scientific approach in analyzing operational decision problems. (These are problems involved in carrying out operations). The way operations research  is the adoption and adaptation of the scientific approach is as follows:

i.  Definition of Problem
    This involves the development of a clear and concise statement of the problem at hand. This gives direction and meaning to other steps.

    In defining the problem, it is important that the whole system be examined critically in order to recognize all the areas that could be affected by any decision taking. It is essential to examine the symptoms and true causes of the problem when defining the issue.

Note that when the problem is difficult to qualify, it may be necessary to develop specific and measurable objectives that may not solve the real problem.

ii. Construction of a Model

This step involves the construction of a suitable model (usually mathematical), which is a representation of the problem at hand.

It might be of a functional nature as in linear programming or have a logical structure as in simulation and algorithms.

E.g. Minimize: $C = 4x + 5y$

Subject to: $x + 3y \geq 6$

$$x + y \geq 3$$

$x, y \geq 0$, which is a linear programming model.

iii. Data Collection

It involves obtaining quantitative data either from existing records or a new survey that fits well into the constructed model of the problem.

iv. Developing a Solution

This involves the manipulation of the model to arrive at the best (optimal) solution to the problem. It may require solving some mathematical equations for optimal decisions as in calculus or linear programming models. It may also be a logical approach or a functional approach which does not require solving a mathematical equation, such as in queuing theory. The optimal solution is then determined by some criteria.

v. Testing the Model and its Solution

This involves determining the accuracy or the completeness of the data used in the model because inaccurate data leads to inaccurate solutions. If the model can adequately predict the effect of the changes in the system, however simple it may be, it is acceptable.

vi. Interpretation of the Results/Establishment of Controls

This involves determining the implication of the solutions to the system. In most cases, a solution to a problem will result in a kind of action or change in the organization. The implication of these actions or changes must be determined and analyzed before results are implemented. This analysis (sensitivity analysis) determines how much the solution changes if there were changes in model or in the input data. Controls are therefore established so that changes that make a significant difference in solution are recognized and taken into account.

vii.  Implementation of Model
    This is the process of incorporating the solution into the system, which is carried out by the personnel already working in the area and not the Operations Research team.

    The solution to the problem has to be translated to a set of operating instructions which can be handled by the personnel involved.

**Flow Chart Showing the Stages in Operations Research**

```
            ┌──────────────────────┐
            │  Problem Definition   │
            └──────────┬───────────┘
                       ▼
            ┌──────────────────────┐
   ┌───────▶│  Model Construction   │◀───────┐
   │        └──────────┬───────────┘        │
   │                   ▼                     │
   │        ┌──────────────────────┐        │
   │        │ Input Data Acquisition│        │
   │        └──────────┬───────────┘        │
   │                   ▼                     │
   │        ┌──────────────────────┐        │
   │        │  Solution Derivation  │        │
   │        └──────────┬───────────┘        │
   │                   ▼                     │
   │        ┌──────────────────────────┐    │
   │        │ Testing of Model and Solution │─┘
   │        └──────────┬───────────────┘
   │                   ▼
   │   ┌────────────────────────────────────┐
   └───│ Analyzing Results/Establishment of Control │
       └──────────┬─────────────────────────┘
                  ▼
       ┌──────────────────────────┐
       │  Implementation of Results │
       └──────────────────────────┘
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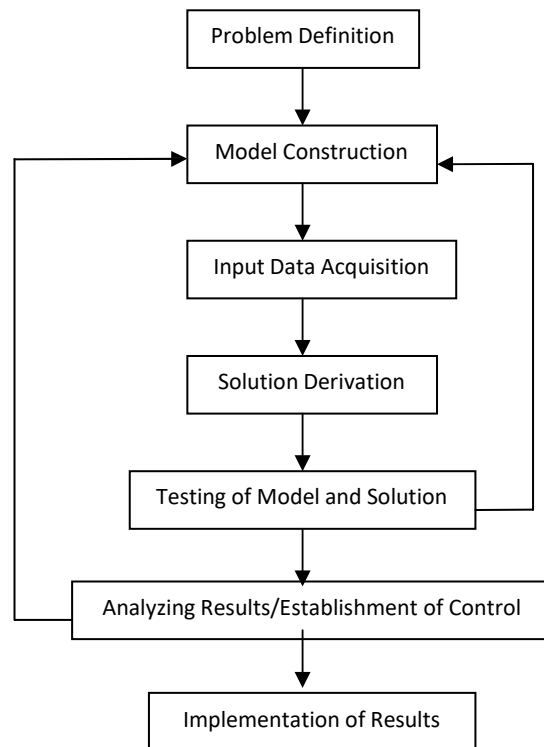
### 3.5 Role of Operations Research in Business

Operations Research help to determines the information requirements that enable the best decisions to be made, subject to cost constraints and to formulation of rules or procedures for making these operating decisions. It is also concern with improving the day-to-day efficiency of a company's operation.

**Business areas where Operations Research is applicable includes**:

i.  Stock re-ordering policies.
ii.  Transport schedules.
iii.  Product mix.

12

iv. Production flows.
v. Allocation problems i.e. which jobs should be allocated to which machines.
vi. Time wasted queuing at issuing, counters.
vii. Scheduling of activities in a complex project.
viii. General congestion problem.

### 3.6 Limitations of Operations Research

i. It is often difficult to balance the requirement of reality and those of simplicity.
ii. The quality of data collection may be poor and/or inaccurate.
iii. In many cases, the solution of Operations Research problem is restricted by the lack of suitable solution techniques.
iv. The derived solution may be sub-optimal i.e. the boundaries of the problem may be open.
v. An Operations Research model is static but the solution it imitates is dynamics.
vi. Conflict between conclusion reached by the Operations Research analyst and the opinion of time managers as to the best course of action.

### 3.7    Use of Computer

The computer as we all know is the major modern in information technology equipment. One of its uses, which most people in our part of the world are not yet attuned to, is problem-solving. Operations research uses the computer in this regard.

The following process is useful in this regard.

i. Identifying situation e.g. production planning.
ii. A number of activities to be performed e.g. tables, chairs, doors to be made. Generalizing term is $j$, the variable representing the magnitude of $jth$ activity is $X_j$.
iii. A number of resources with general term $i$ and available quantity $b_i$.
iv. Profit yield per unit of $jth$ activity - $C_i$.
v. Amount of $ith$ resource required by 1 unit of $jth$ activity - $a_{ij}$.
vi. Obtain the model:
The logic of the situation can be studies to achieve the famous linear programming model:

Maximize: $Z = \sum C_j X_j$

Subject to: $\sum a_{ij} X_j \le b_i , v_i$

$$X_j \ge 0.$$

vii.  Solve the model by using the computer that has software for solving linear programming model of our size. The computer, after keying the data will give the solution.

**EXAMPLE (A Break Even Model)**

The sales manager of Turnover Limited maintains he could increase the sales turnover (in units) of any of the company's product by 50 percent if he was authorized to give a  10%  price discount and place appropriate additional advertising matter.

The Board wish to know the maximum additional advertising expense they can incur in respect of any given product without the manager's proposal resulting in a smaller profit.

**SOLUTION**

This situation can be modeled as follows:

Let *p* be the current production selling price, *m* be the product marginal cost and, *q* be the current turnover (in units).

The current product contribution is: $q(p - m)$
and

the new product contribution net of the additional advertising is:

$1.5q(0.9p - m) -$ cost of additional advertising.

Since at the worst, these two contributions must equal each other to make proposal viable, then:

$q(p - m) = 1.5q(0.9p - m) -$ cost of additional advertising.

Cost of additional advertising is:

$1.5q(0.9p - m) - q(p - m) = 1.35qp - 1.5qm - qp + qm$

$$= q(0.35p - 0.5m).$$

Since we can put this expression in a more useful form by taking $1/2$ out of the brackets, we have:

Maximum additional advertising cost to be:

14

$$\frac{n}{2}(0.7 \times 10 - 4).$$

The Board now have a useful model, for if one of their products has a marginal cost #4 and sells 10,000 units at #10, then:

Maximum additional product advertising cost is:

$$\frac{10,000}{2}(0.7 \times 10 - 4) = \#15,000.$$

The model, however, goes further than this, for it also tells the Board that any product with a marginal cost of 70% or more of the selling price must never be subjected to the sales manager's proposal. This follows from the fact that if $M > 0.7p$, a negative result i.e. the additional advertising would need to be negative. It would, then, be impossible to allow the manager any additional advertising expenditure.

## 4.0 CONCLUSION

This unit introduced you to the formal study of Operations Research. However, what has been discussed is not exhaustive. Further reading is recommended.

## 5.0 SUMMARY

- Operations Research is scientific in nature and is closely related to management science.
- Operations Research has some limitations
- Effective use of Operations Research involves people working as team to solve identified problems

## 6.0 TUTOR MARKED ASSIGNMENT

(1). Explain the term Operations Research

(2). Describe the role, methodology and limitations of Operations

Research

## 7.0 REFRENCES/FURTHER READING

1.    Operations Research. By Prem Kumar Gupta and D.S. Hira

2.    Operations Research. ( Problem Solvers) By Research and

      Education  Association.

3.    Statistics and Operations Research. By Debashis Dutta

4.    Operations Research by  O.T. Arowolo and G.O. Lawal

## UNIT 2    MODELLING IN OPERATIONS RESEARCH

**CONTENTS**

## 1.0    INTRODUCTION

Modeling is fundamental to the Operations Research approach.

It is a way of representing the behaviour of a situation to enable us deduce what is best to do about the system. Models are therefore tools for representing a situation to understand it and for reasoning about it.

If for instance, the relationship between the demand for an item and time is found to be represent able by a seasonal pattern, then we can use the representation to identify when it is best for us to trade in the item. This is the kind of thing operations research disciplines us to do as an objective and rational way of tackling situations.

The model we build must be based on the facts of the situation being modeled, only being represented in a convenient way either for better comprehension or for reasoning with or to communicate with. The facts are usually in terms of:

i.   What the under acting entities are e.g. item and time in the above example.
ii.  How the entity interrelate e.g. seasonal relationship between the item and time in the above example, in which relationship we call time series.

A major merit in modelling is that knowledge gain about a model framework can be used in understand, communicating, and/or reasoning about other situation that can be modelled using the same framework. Hence, for instance knowledge generally gain about network model framework is adaptable for me in studying and analyzing projects, decision networks, road network, etc.

This approach is a generalizing one and therefore, it is efficient.

**2.0.  OBJECTIVES**
At the end of this unit you will be able to :

- State the meaning of model in Operations Research
- Describe the various types of model
- Describe how to construct a model
- State some standard  Operations Research model

**3.0 .  MAIN TEXT**

**3.1   <u>Types of Models</u>**

1.      **Symbolic or Mathematical Model**

This is the most important type of model. Mathematical modeling  focuses on creating a mathematical representation of management problems in organizations. All the variables in a particular problem are expressed mathematically. The model then provides different  outcomes, which will result from the different choices the management wishes to use. The best outcome in a particular situation will help the management in decision-making. These models use set of mathematical symbols and hence are also called symbolic models.

The variables in many business and industry situations can be related together by mathematical equations. To understand the concepts of symbolic or mathematical model, visualize a balance sheet or profit and loss account as a symbolic representation of the budget. Similarly, the demand curve in economics can be seen as symbolic representation of the buyers' behaviour at varying price levels.

**2.     Simulation Models**

In Simulation model, the behaviour of the system under study is 'initiated' over a period of time'. Simulation models do not need mathematical variables to be related in the form of equations, normally, these models are used for solving such problems that cannot be solved mathematically. Simulation is a general technique, which helps us in developing dynamic models, which are similar to the real process. Developing good simulation models is difficult because 'creating' a real life situation to perfection is extremely difficult.

**3.     Iconic Models**

These models represent the physical simulations to the real life system under. Physical dimensions are scaled up or down to simplify the actual characteristic and specifications of the system. Preparation of prototype models for say an automobile or 3-D plant layout are some examples of iconic models.

**4.     Analog Model**

They are not the exact replica. Like the iconic models these are smaller, simple physical system as compared to the real life system which are complex. These models are used to explain an actual system by analog.

**5.     Deterministic Model**

When the change of one variable has a certain or define change in the outcome, the model is called a deterministic model. Infact, everything is absolutely clearly defined and the results are known. Economic Order Quantity (EOQ) is a deterministic model, as economic lot size can be exactly known, with change in one of the variables in the EOQ formula.

**3.2     Types of Mathematical Models**

Mathematical models can be classified in several ways. These include the following:

**Probabilistic versus Deterministic**

Models that incorporate uncertainty in terms of probabilities of future event occurring are probabilistic ones, while those that assume certain knowledge of such events are deterministic ones.

Deterministic models predict the exact outcome of a situation because it is based on certain known laws.

Probabilistic models deals with situation that are random in character and can predict the outcome within a certain stated or known degree of accuracy.

For example, in a project management situation, the critical path method ($CPM$) uses a deterministic network model in which the durations of the activities are assumed to be known for certain. Whereas, the program evaluation and review technique ($PERT$) uses a model that incorporates some probability distribution of the durations.

**Qualitative Versus Quantitative**

Mathematical models are essentially symbolic and structural. Where the entities involved are kinds of things or attributes rather than the quantities in which they exist, then we have a qualitative model.

Quantitative models, on the other hand involve quantities of attributes as the interacting entities.

Qualitative (or conceptual) models are often starting points in formulating quantitative models. Inventory and linear programming mathematical models involve such qualitative frameworks as starting points.

**Linear versus Non-linear**

A mathematical model is normally put in the form of relationships between the quantitative variables. The relationships could be linear or non-linear. They are linear when changes in the independent or input variables result in constant proportional changes in the dependent, or output variables.

The graphs of such relationships are in the form of straight lines. The linearity property is one of constant return to scale.

Examples of linear relationship are:

$y = 4x + 2$

$y = 5 + 5w + 3x$

While examples of non-linear relationships are:

$y = 6 + 5x + x^2$

$y = xz^2 + 3x^2 + 5z + 2$

**Static versus Dynamic**

Models that assume that the situation being represented will not change, at least in its essential features, within the operational period, are static ones. While those that incorporate changes are dynamic ones. In this respect, linear programming models that use only one set of values of the input variables are static, while dynamic programming that use only one set of values of the input variables that change over time are dynamic.

**Standard versus Custom-made Models**

Standard models are format for representing recurring features and relationships. They are usually put in generalized forms. They are used in appropriate specific situations by replacing the features in the models with the corresponding features in the real, particular situation. There are several such models that are in use in operations research. Linear programming models are an example of standard models.

Custom-made models, on the other hand are obtained specially for specific situations. This has to be done when no standard model framework exists, or is known to be reasonably appropriate for the specific situation. We refer to such modelling as being from fundamentals.

**Analytic versus Simulation**

An analytic model is one that represents the relationship between the variables in the form of formulas. Linear programming model is an example.

A simulation model, on the other hand, describes the process involved in a simulation, indicating the mathematical relationships that exist at each stage.

In this sense, simulation is an imitation of the step-by-step process involved in the build-up of system relationships. We shall return to the subject of simulation later.

**3.3 Structure of Operations Research Mathematical Models**

Operations Research Mathematical Models are decision problem models in the following general form:

| Entity | Representation |
|--------|----------------|
| Objective | $E$ |
| Factors: | |
| 1. Controllable (or decision) variables | $X_i$ |
| 2. Uncontrollable variables | $Y_i$ |
| Structure | $E = f(X_i, Y_i)$ |

Each model will have entities in them that are in the foregoing categories.

In a simple inventory situation for example, the interacting entities are the following:

| Entity | Symbol |
|---|---|
| Total cost | $TC$ |
| Order quantity | $Q$ |
| Purchase cost | $C_p$ |
| Ordering cost | $C_o$ |
| Holding cost | $C_h$ |
| Demand | $D$ |

The conceptual model can then be put into symbolic (mathematical) form as follows:

$$TC = DC_p + \frac{D}{q}C_o + qC_p C_h.$$

Thus, we see that

$X_i$ ⟶ $Q$

$Y_j$ ⟶ $D, C_p, C_o, C_n$
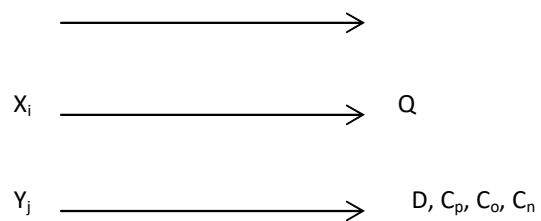
Solving a mathematical model involves deducing what values of the decision (controllable) variables will yield best result.

We can apply appropriate pure mathematics method to make such deductions. In the case of the simple inventory model, the method of calculus is used in solving optimization problem involved to yield the famous Economic Order Quantity (*EOQ*) formula:

$$Q = \sqrt{\frac{2DC_o}{C_p C_h}} \; .$$

### 3.4 **How to Construct a Model**

To construct a model. You need to observe real-word problem and make assumptions about influencing factors. This is called abstraction.

```
( Real-word Problem )   ⟹   ( Mathematical )
```
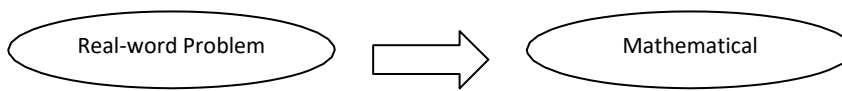
You must know enough about the mechanics of mathematics to derive results from model.

```
      ( Mathematical )
            │
        Derive
        Result
            ▼
       ( Predictions )
```
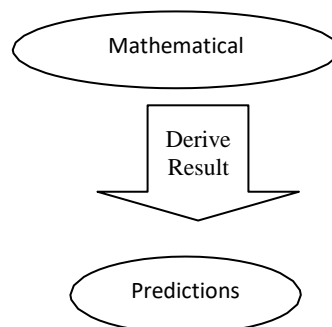
The next step is to gather data. Does the prediction given by the model fit all the known data? If not, you will use the data to modify the assumptions used to create the model. This is an ongoing process.
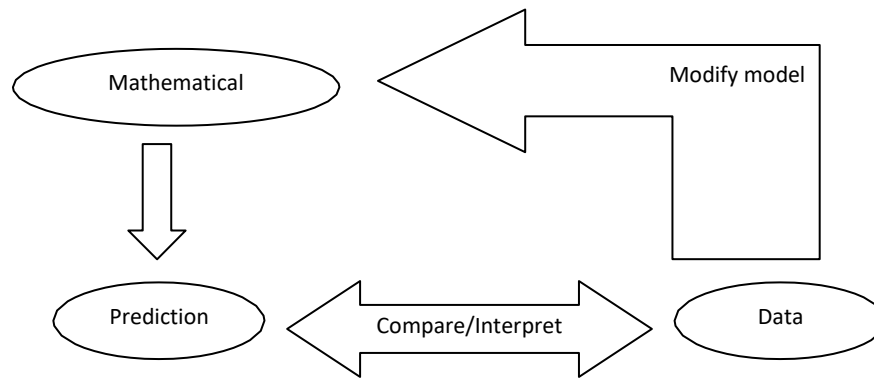
### 3.5 <u>Standard Operations Research Models</u>

Operations Research practice over the years has led to the identification of problems that recur in diverse situations. Ways of representing (modelling) them have been developed and techniques for solving them also developed. These problem types are called prototypes. While models for representing them bear specific names and so also are the techniques for solving them.

The table below summarizes the common prototypes, their nature, model and techniques for handling them.

Major prototypes and common techniques available for solving them.

| Prototype | Nature | Common Solution Technique |
|---|---|---|
| Allocation and its extensions e.g. Transportation | How best to allocate limited resources to activities | 1. Mathematical programming e.g. Linear programming, integer programming, goal programming.<br>2. Specific techniques e.g. Transportation Techniques, Assignment Techniques. |
| Inventory | Determining optimal order quantity in a situation requiring purchase to store pending usage | 1. Inventory theory techniques e.g. the ($EOQ$) method.<br>2. Simulation. |
| Waiting lines | What best level of facilities will minimize cost of queues, or what order of service will do so | 1. Queuing theory<br>2. Simulation |
| Coordination and Extension | How best to manage pre-set sequencing of jobs or activities e.g. comprising a projector an assembly line | 1. Network techniques e.g. *CPM* and *PERT*.<br>2. Line balancing techniques. |
| Replacement; | Maintenance of operating units | 1. Replacement techniques |

| | | |
|---|---|---|
| including general maintenance | through replacement and/or servicing | 2. Maintenance techniques<br>3. Simulation<br>4. Manpower management models |
| Competition | What strategy is best to adopt in a conflict e.g. competitive situation | 1. Game theory<br>2. Hyper games<br>3. Meta games |

## 4.0. CONCLUSION

Mathematical Model are the most used operations research model, they help to simplify complex situation for ease of understanding and evolving solution to problems.

## 5.0. SUMMARY

* Models help to simplify complex situation for ease of

understanding and evolving solution to problems.

* The component of an operations research model are variable and

Parameter.

* Mathematical Model are the widely use operations research model

## 6.0. TUTOR MARKED ASSIGNMENT

1. (a) Explain the term models in operations research

(b) Enumerate the types of models used in operations research.

2. In a particular one product company, it has been noted that the value

of sales per day is given by the expression : $\{\#1000 - sellingpriceperunitin\#\}$ If daily fixed cost amount to #1000

and the variable cost per unit is #2, then, prepare a profit model

## 7.0. References/ Further Readings

1.      Operations Research. By Prem Kumar Gupta and D.S. Hira

2.      Operations Research. ( Problem Solvers) By Research and

        Education  Association.

5.      Statistics and Operations Research. By Debashis Dutta

6.      Arowolo .O. (2011) Operations Research: CIT656

7.      Operations Research by  O.T. Arowolo and G.O. Lawal

**MODULE TWO  MATHEMATICAL PROGRAMMING**

UNIT 1:          Optimization problem

UNIT 2:          Linear programming (LP)

UNIT 3:          Linear Programming (Simplex Method)

UNIT 4:          INTEGER PROGRAMMING


**UNIT 1: OPTIMIZATION PROBLEMS**

1.0      Introduction

2.0.     Objective

3.0.     Main Content

          3.1 Linear Programs

          3.2. Integer Programs

          3.3. Quadratic Programs

          3.4. Problem Formulation

          3.5. Solved Problems

4.0. Conclusion

5.0. Summary

6.0. Tutor Marked Assignment

7.0.Reference/ Further Readings


**UNIT 1: OPTIMIZATION PROBLEMS**


**1.0      Introduction**

In an *optimization problem* one seeks to maximize or minimize a specific quantity, called the *objective,* which depends on a finite number of input variables. These variables may be independent *of* one another, or they may be related through one or more *constraints*.

**Example 1.1** The problem

$$\text{minimize:} \quad z = x_1 + x_2$$

$$\text{subject to:} \quad x_1 - x_2 = 3$$

$$x_2 > 2$$

is an optimization problem for the objective z. The input variables are $x_1$ and x2, which are constrained in two ways: x, must exceed $x_2$ by 3, and also $x_2$ must be greater than or equal to 2. It is desired to find values for the input variables which minimize the sum of their squares, subject to the limitations imposed by the constraints.

A *mathematical program is* an optimization problem in which the objective and constraints are given as mathematical functions and functional relationships (as they are in Example 1.1). Mathematical programs treated in this book have the form

$$\text{optimize:} \quad z = f(X_1, x_2, \ldots, X_n)$$

$$\text{subject to:} \quad g_1(x_1, x_2, \ldots, X_n) \quad b_i$$

$$g_2(X_1, X_2, \quad X_n) \quad b_2$$

$$\ldots\ldots\ldots\ldots \_ \ldots \quad (1.1)$$

$$g_m(X_1, X_2, \quad X_n) \quad b_m$$

Each *of* the *m* constraint relationships in (1.1) involves one of the three signs $\leq$, $=$, $\geq$. *Unconstrained* mathematical programs are covered by the formalism (1.1) if each function $g_i$ *is* chosen as zero and each constant $b_i$ *is* chosen as zero.

## 2.0. Objective

✓ Understand Optimization problems

✓ Understand Linear and Non Linear Programs

✓ Formulate Problems from verbal statements.

✓ Differentiate between Integer and Quadratic programs

✓ Understand solution conventions in OR

## 3.0. Main Content

## 3.1. LINEAR PROGRAMS

A mathematical program (1.1) *is linear if* $f(x_1, x_2, \ldots, x_n)$ and each $g_i(x_1, x_2, \ldots x_n)$ $(i = 1, 2, \ldots, m)$ are linear in each of their arguments-that is, if

$$f(X_1, X_2 \ldots, x_n) = c_1 X_1 + c_2 X_2 + \ldots + c_n X_n \qquad (1.2)$$

and

$$9_i(X_1, X_2 \ldots, X_n) = a_{i1} x_1 + a_{i2} X_2 + \ldots + a_{in} X_n \qquad (1.3)$$

where $c_j$ and *aid* $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ are known constants.

Any other mathematical program is *nonlinear. Thus,* Example 1.1 describes a nonlinear program, in view of the form of z.

## 3.2. INTEGER PROGRAMS

An *integer program* is a linear program with the additional restriction that the input variables be integers. It is not necessary that the coefficients in (1.2) and (1.3), and the constants in (1.1), also be integers, but this will very often be the case.

## 3.3. QUADRATIC PROGRAMS

A *quadratic program* is a mathematical program in which each constraint is linear-that is, each constraint function has the form (1.3)-but the objective is of the form

$$f(x_1, x_2, \ldots, x_n) = Y\_ \qquad Y\_ c_{ii} x_i x_i + I\, d_i x_i \qquad (1.4)$$

$$i=1 \ i=1 \qquad i=1$$

Where $c_{ij}$ and $d_i$ are known constants.

The program given in Example 1.1 is quadratic. Both constraints are linear, and the objective has the form (1.4), with n = 2 (two variables), $c_{11} = 1$, $c_{12} = c_{21} = 0$, $c_{22} = 1$, and $d_1 = d_2 = 0$.

## 3.4. PROBLEM FORMULATION

Optimization problems most often are stated verbally. The solution procedure is to model the problem with a mathematical program and then solve the program by the techniques described in Chapters 2 through 15. The following approach is recommended for transforming a word problem into a mathematical program:

*STEP 1:*      Determine the quantity to be optimized and express it as a mathematical function. Doing so serves to define the input variables.

*STEP* 2:      Identify all stipulated requirements, restrictions, and limitations, and express them mathematically. These requirements constitute the constraints.

*STEP* 3:    Express any hidden conditions. Such conditions are not stipulated explicitly in the problem but are apparent from the physical situation being modeled. Generally they involve nonnegativity or integer requirements on the input variables.

## 3.5. SOLUTION CONVENTION

In any mathematical program, we seek *a* solution. If a number of equally optimal solutions exist, then any one will do. *There is no preference between equally optimal solutions if there is no preference stipulated in the constraints.*

## 3.6. SOLVED PROBLEMS

3.6.1  The Village Butcher Shop traditionally makes its meat loaf from a combination of lean ground beef and ground pork. The ground beef contains 80 percent meat and 20 percent fat, and costs the shop 80~ per pound; the ground pork contains 68 percent meat and 32 percent fat, and costs 60~ per pound. How much of each kind of meat should the shop use in each pound of meat loaf if it wants to minimize its cost and to keep the fat content of the meat loaf to no more than 25 percent?

The objective is to minimize the cost (in cents), z, of a pound of meat loaf, where

z = 80 times the poundage of ground beef used plus 60 times the poundage of ground pork used Defining

$x_1$ = poundage of ground beef used in each pound of meat loaf

$x_2$ - poundage of ground pork used in each pound of meat loaf we express the objective as

   minimize:    $z = 80x_1 + 60x_2$

Each pound of meat loaf will contain $0.20x_1$ pound of fat contributed from the beef and $0.32x_2$ pound of fat contributed from the pork. The total fat content of a pound of meat loaf must be no greater than 0.25 lb. Therefore,

$$0.20x_1 + 0.32x_2 < 0.25 \qquad (2)$$

The poundages of beef and pork used in each pound of meat loaf must sum to 1; hence

   $x, + x_2 = 1$                                        (3)

Finally, the butcher shop may not use negative quantities of either meat, so that two hidden constraints are x, $>\_$ 0 and $x_2 >\_$ 0. Combining these conditions with (1), (2), and (3), we obtain

minimize:    $z = 80x, + 60x2$

subject to:    $0.20x_1 + 0.32x_2 < 0.25$                     (4)

 $x,+ x_2=1$

with:   all variables nonnegative

System (4) is a linear program. As there are only two variables, a graphical solution may be given.

3.6.2   Solve the linear program (4) of Problem 1.1 graphically.

See Fig. 1-1. The *feasible* region-the set of points $(x_1, x_2)$ satisfying all the constraints, including the nonnegativity conditions-is the heavy line segment in the figure. To determine $z^*$, the minimal value of z, we arbitrarily choose values of z and plot the graphs of the associated objectives. By choosing $z = 70$ and then $z = 75$, we obtain the objectives

$$70 = 80x_1 + 60x_2 \quad \text{and} \quad 75 = 80x_1 + 60x_2$$

respectively. Their graphs are the dashed lines in Fig. 1-1. It is seen that $z^*$ will be assumed at the upper endpoint of the feasible segment, which is the intersection of the two lines

$$0.20x_1 + 0.32x_2 = 0.25 \quad \text{and} \quad x_1 + x_2 = 1 \ r,$$

Simultaneous solution of these equations gives $x_1 = 7/12$, $x2 = 5/12$; hence,

$$z^* = 80(7/12) + 60(5/12) = 71.67$$

3.6.3   A furniture maker has 6 units of wood and 28 h of free time, in which he will make decorative screens. Two models have sold well in the past, so he will restrict himself to those two. He estimates that model I requires 2 units of wood and 7 h of time, while model II requires 1 unit of wood and 8 h of time. The prices of the models are $120 and $80, respectively. How many screens of each model should the furniture maker assemble if he wishes to maximize his sales revenue?

The objective is to maximize revenue (in dollars), which we denote as z:

z = 120 times the number of model I screens produced plus 80

times the number of model II screens produced

Letting

$x_1$ - number of model I screens to be produced

$x_2$ - number of model II screens to be produced

we express the objective as

maximize:     $z = 120x_1 + 80x_2$                                          (1)

The furniture maker is subject to a wood constraint. As each model I requires 2 units of wood, $2x_1$ units must be allocated to them; likewise, $1x_2$ units of wood must be allocated to the model 11 screens. Hence the wood constraint is

31

$$2x, + x_2 < 6 \qquad\qquad\qquad\qquad (2)$$

The furniture maker also has a time constraint. The model I screens will consume $7x,$ hours and the model II screens $8x_2$ hours; and so

$$7x, + 8x_2 < 28 \qquad\qquad\qquad\qquad (3)$$

It is obvious that negative quantities of either screen cannot be produced, so two hidden constraints are $x, >_{-} 0$ and $x_2 >_{-} 0$. Furthermore, since there is no revenue derived from partially completed screens, another hidden condition is that $x,$ and $x_2$ be integers. Combining these hidden conditions with (1), (2), and (3), we obtain the mathematical program

> maximize: $\quad z = 120x, + 80x2$
>
> subject to: $\quad 2x, + x_2 < 6 \qquad\qquad\qquad\qquad (4)$
>
> $7x, + 8x_2 < 28$
>
> with: all variables nonnegative and integral

System (4) is an integer program. As there are only two variables, a graphical solution may be given.

3.6.4 Give a graphical solution of the integer program (4) of Problem 1.3.

See Fig. 1-2. The feasible region is the set of integer points (marked by crosses) within the shaded area. The dashed lines are the graphs of the objective function when z is arbitrarily given the values 240, 330, and 380. It is seen that the z-line through the point (3, 0) will furnish the desired maximum; thus, the furniture maker should assemble three model I screens and no model II screens, for a maximum revenue of

$$z^* = 120(3) + 80(0) = \$360$$

Observe that this optimal answer is not achieved by first solving the associated linear program (the same problem without the integer constraints) and then moving to the closest feasible integer point. In fact, the feasible region for the associated linear program is the shaded area of Fig. 1-2; so the optimal solution occurs at the circled corner point. But at the closest feasible integer point, (2, 1), the objective function has the value $z = 120(2) + 80(1) = \$320$ or \$40 less than the true optimum.

An alternate solution procedure for Problem 1.3 * given in Problem 7.8.

3.6.5 Universal Mines Inc. operates three mines in West Virginia. The ore from each mine is separated into two grades before it is shipped; the daily production capacities of the mines, as well as their daily operating costs, are as follows:

|  | High-Grade Ore, tons/day | Low-Grade Ore, tons/day | Operating Cost, \$1000/day |
|---|---|---|---|
| Mine I | 4 | 4 | 20 |
| Mine II | 6 | 4 | 22 |

| Mine III | 1 | 6 | 18 |
|----------|---|---|----|

Universal has committed itself to deliver 54 tons of high-grade ore and 65 tons of low-grade ore by the end of the week. It also has labor contracts that guarantee employees in each mine a full day's pay for each day or fraction of a day the mine is open. Determine the number of days each mine should be operated during the upcoming week if Universal Mines is to fulfill its commitment at minimum total cost.

Let $x_1$, $x_2$, and $x_3$, respectively, denote the numbers of days that mines I, 11, and III will be operated during the upcoming week. Then the objective (measured in units of $1000) is

minimize:  $z = 20x_1 + 22x_2 + 18x_3$                      (1)

The high-grade ore requirement is

$$4x_1 + 6x_2 + x_3 > 54 \qquad (2)$$

and the low-grade ore requirement is

$$4x_1 + 4x_2 + 6x_3 \geq 65 \qquad (3)$$

As no mine may operate a negative number of days, three hidden constraints are $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$. Moreover, as no mine may operate more than 7 days in a week, three other hidden constraints are $x_1 < 7$, $x_2 < 7$, and $x_3 < 7$. Finally, in view of the labor contracts, Universal Mines has nothing to gain in operating a mine for part of a day; consequently, $X1$, $X2$, and $X3$ are required to be integral. Combining the hidden conditions with (1), (2) and (3), we obtain the mathematical programme

minimize  $z = 20x1 + 22x2 + 18x3$

subject to:  $4x1 + 6x2 + x3 \geq 54$

$4x1 + 4x2 + 6x3 \geq 65$

$x1 \qquad\qquad \leq 7$

$\qquad x2 \qquad \leq 7$

$\qquad\qquad x3 \quad \leq 7$

with: all variable nonnegative and integral

## 4.0. Conclusion

In an *optimization problem* one seeks to maximize or minimize a specific quantity, called the *objective,* which depends on a finite number of input variables. These variables may be independent *of* one another, or they may be related through one or more *constraints*.

## 5.0. Summary

In any mathematical program, we seek *a* solution. If a number of equally optimal solutions exist, then any one will do. *There is no preference between equally optimal solutions if there is no preference stipulated in the constraints.*

## 6.0. Tutor Marked Assignment

1. Minimize $z = 20x_1 + 22x_2 + 18x_3$

      Subject to:     $4x_1 + 6x_2 + x_3 \geq 54$

                 $4x_1 + 4x_2 + 6x_3 \geq 65$

                 $x_1 \qquad\qquad \leq 7$

                      $x_2 \qquad \leq 7$

                          $x_3 \quad \leq 7$

      With: all variable nonnegative and integral

## 7.0. REFERENCES/FURTHER READINGS

1.      Operations Research. By Prem Kumar Gupta and D.S. Hira

2.      Operations Research. ( Problem Solvers) By Research and

     Education  Association.

8.      Statistics and Operations Research. By Debashis Dutta

9.      Operations Research by  O.T. Arowolo and G.O. Lawal

10.      Operations Research (1997) Richard Bronson & Govindasami Naadimuthu

     (Schaum series)

## UNIT 2      LINEAR PROGRAMMING (LP)

**1.0 Introduction**
2.0     Objective
3.0     Main Content
       3.1     Application of LP to Business
       3.2     Properties of LP Model
       3.3     Assumption of LP
       3.4     Formulation of LP Model
       3.5     Examples on Formulation of LP Model
       3.6     Graphical Solution of Linear Programming Problems

## 1.0.    INTRODUCTION

Many management decisions involve trying to make the most effective use of organization resources. These resources include Machinery, Labours, Money, Time, Warehouse space or Raw materials to produce goods (machinery, furniture, food or cooking) or service (schedules for machinery and production advertising policies or investment decision).

Linear Programming (LP) is a widely used mathematical techniques designed to help managers in planning and decision making relative to resource allocations.

## 2.0.    OBJECTIVE

By the end of this unit you should be able to :

- State the usefulness of linear programming in operations research
- State the properties of a linear programming model
- Identify some areas of application of linear programming
- Formulate a linear programming model
- State the usual assumptions of a linear programming model
- Solve a two-variable  linear programming model graphically

## 3.0.    MAIN TEXT

### 3.1.    Application of Linear Programming to Business

i.  Product-Mix
**Use in the** selection of the product-mix in a factory to make best use of machine and machine hours available while maximizing profit, that is, to find out which product to include in production plan and in what quantities that should be produced.

ii.  Blending Problems
Use for the selection of different blends of raw materials to produce the best combinations at minimum cost e.g. food drinks, etc.

iii.  Production Scheduled

Use to develop a production scheduled that will satisfy future demands for a firm's product and at the same time minimize production and inventory cost.

**iv.** Production Quantity
Use in determination of how much quantity to produce of different grades of petroleum product (say) to yield maximum profit.

**v.** Distribution System
Use in determining a distribution system that will minimize total shipping cost from several warehouses to various market locations.

**vi.** Limited Advertisement
Use in the allocation of limited advertising budget among radio, TV and newspaper spots in order to maximize the returns on investment.

**vii.** Investment
Use in selecting investment port-folio from a variety of stocks and bonds available in such a way as to maximize the returns on investment.

**viii.** Work Scheduled
Use in the development of a work scheduled that allows a large restaurant to meet staff needs at all hours of the day, while minimizing the total number of employees.

## 3.2 Properties of Linear Programming Model

All linear programming models have four basic properties in common. They are:

i. All LP models seek to maximize or minimize some quantity, usually profit or costs.
ii. All LP models have constraints or limitations that limit the degree to which the objective can be purse. E.g. deciding how many units of product in a product line to be produced is restricted to the manpower and machinery available.
iii. There must be alternative course of action to choose from e.g. if there are 4 different product, management may decide (using LP) how to allocate limited resources among them.
iv. Objectives and constraints in LP model must be expressed in linear equations and inequalities.

## 3.3   Assumption of Linear Programming

  i.  Certainty
    We assume that numbers in the objective and constraints are known with certainty and does not change during the period being study

  ii.  Proportionality
    We are sure that proportionality exists in the objective and the constraints. This means that, if production of one unit of product uses two of a particular scare resource; then making five units of that product uses ten resources.

  iii.  Additivity
    This means that the total of all activities equals the sum of each individual activity.

  iv.  Divisibility
    This means that solution may take fractional values and need not be in whole numbers (integers). If a fraction of a product cannot be produced, integer programming problem exist.

  v.  Non-negativity
    We assume that all answers or variables are non-negative. Negative values of physical quantities are an impossible solution.

## 3.4   Formulation of Linear Programming Model

Linear programming problems are optimization problem which are stated verbally or in words. The following steps can be used to formulate the model of any optimization problem.

Step 1

Choose variables and notations that will be used to form the objective and constraints functions.

Step 2

Identify the objective function to either maximize or minimize e.g. maximizes profit or minimizes cost.

Step 3

Develop mathematical relationships to describe objective and constraints.

### 3.5     Examples on Formulation Linear Programming Models

**EXAMPLE 1 (Allocation of Resources in Production)**

A farmer has $100$ acres on which to plant two crops: corn or wheat. To produce these crops, there are certain expenses as shown in the table.

| Item | Cost per Acre (#) |
|---|---|
| **Corn** | |
| Seed | 12 |
| Fertilizer | 58 |
| Planting/care/harvesting | 50 |
| Total | 120 |
| **Wheat** | |
| Seed | 40 |
| Fertilizer | 80 |
| Planting/care/harvesting | 90 |
| Total | 210 |

After the harvest, the farmer must store the crops awaiting proper market conditions. Each acre yields an average of $110$ bushels of corn or $30$ bushels of wheat. The limitations of resources are as follows:

Available capital:                #15,000 .

Available storage facilities:    4,000 bushels.

If net profit (the profit after all expenses have been subtracted) per bushel of corn is #1.30 and for wheat is #2.00, how should the farmer plant the 100 acres to maximize the profits?

## SOLUTION

To formulate a mathematical model, begin by letting:

$x$ = Number of acres to be planted in corn.

$y$ = Number of acres to be planted in wheat.

There are certain limitations or constraints.

- The number of acres planted cannot be negative, so
  $x \geq 0$

  $y \geq 0$

- The amount of available land is 100 acres:
  $x + y \leq 100$.

  Why not $x + y = 100$? it might be more profitable for the

  farmer to leave some land out of production i.e. it is not

  necessary to plant all the land.

- We also know that:
  Expenses for planting the corn $= 120x$.

  Expenses for planting the wheat $= 210\,y$.

  The total expenses cannot exceed #15,000:

  This is the available capital:

  $120x + 210\,y \leq 15000$

- The yields are:
  Yield of acreage planted in corn $= 110x$.

  Yield of acreage planted wheat $= 30\,y$.

  The total yield cannot exceed the storage capacity of

  4,000 bushels:

$110x + 30y \leq 4000$.

Summary of constraints:

| | |
|---|---|
| $x + y \leq 100$ | [Available land] |
| $120x + 210y \leq 15000$ | [Available capital] |
| $110x + 30y \leq 4000$ | [Storage capacity] |
| $x \geq 0$ | [Non-negativity] |
| $y \geq 0$ | [Non-negativity] |

- Now, let P represent total profit. The farmer wants to maximize the profit, P.
  Profit from corn $=$ value . amount

$$= 1.30 \ . \ 110x$$

$$= 143x \ .$$

Profit from wheat $=$ value . amount

$$= 2.00 \ . \ 30y$$

$$= 60y \ .$$

P $=$ profit from corn $+$ profit from wheat $= 143x + 60y$ .


The linear programming model is stated as follows:

Maximize:  $P = 143x + 60y$          [Available land]

Subject to:

| | |
|---|---|
| $x + y \leq 100$ | [Available land] |
| $120x + 210y \leq 15000$ | [Available capital] |
| $110x + 30y \leq 4000$ | [Storage capacity] |
| $x \geq 0$ | [Non-negativity] |
| $y \geq 0$ | [Non-negativity] |

**EXAMPLE 2 (Allocation of Resources in Manufacturing)**

ONIJOGBO manufactures two types of settee; half-upholstered and full-upholstered. The contribution per unit to profit is #80 for half-upholstered and #90 for full-upholstered. The amount of materials needed per product and maximum available materials are given below:

| Product | Unit of Material | | |
|---|---|---|---|
| | Wood | Foam | Cover |
| Half-upholstered | 2 | 2 | 5 |
| Full-upholstered | 1 | 4 | 5 |
| Maximum available | 12 | 24 | 35 |

Required: formulate the linear programming model for the above problem.

**SOLUTION**

We want to maximize the profit, P. There are two types of items, half-upholstered and full-upholstered.

Let, $x$ = Number of half-upholstered produced.

$y$ = Number of full-upholstered produced.

Then,

Profit from half-upholstered $= \#80x$.

Profit from full-upholstered $= \#90y$.

P = profit from half-upholstered + profit from full-upholstered
$P = \#80x + \#90y$.

The constraints are:

i. <u>Non-negativity</u>
$x \geq 0$

41

$y \geq 0$

The number of product must be non-negative.

ii. Wood Material

$2x = $ Amount of wood material used for half-upholstered.

$y = $ Amount of wood material used for full-upholstered.

The total wood material cannot exceed $12$:

This is the maximum available: $2x + y \leq 12$.

iii. Foam Material

$2x = $ Amount of foam material used for half-upholstered.

$4y = $ Amount of foam material used for full-upholstered.

The total foam material cannot exceed $24$:

This is the maximum available: $2x + 4y \leq 24$.

iv. Cover Material

$5x = $ Amount of cover material used for half-upholstered.

$5y = $ Amount of cover material used for full-upholstered.

The total cover material cannot exceed $35$:

This is the maximum available: $5x + 5y \leq 35$.

Thus, the linear programming model is:

Maximize: $P = \#80x + \#90y$.

Subject to:

$$2x + y \leq 12 \qquad \text{[Wood material]}$$

$$2x + 4y \leq 24 \qquad \text{[Foam material]}$$

$$5x + 5y \leq 35 \qquad \text{[Cover material]}$$

$$x \geq 0 \qquad \text{[Non-negativity]}$$

$$y \geq 0 \qquad \text{[Non-negativity]}$$

**EXAMPLE 3 (Diet Problem)**

A convalescent hospital wishes to provide at a minimum cost, a diet that has a minimum of $200g$ of carbohydrates, $100g$ of protein and $120g$ of fats per day. These requirements can be met with two foods:

| Food | Carbohydrates | Protein | Fats |
|------|---------------|---------|------|
| A | $10g$ | $2g$ | $3g$ |
| B | $5g$ | $5g$ | $4g$ |

If food A cost $29k$ per ounce and food B cost $15k$ per ounce, how many ounces of each food should be purchased for each patient per day in order to meet the minimum requirements at the lowest cost?

Required: formulate the LP model.

**SOLUTION**

Let, $x$ = Number of ounces of food A.

$y$ = Number of ounces of food B.

The minimum cost, C, is found by:

Cost of food A $= .29x$.

Cost of food B $= .15y$.

$\therefore C = .29x + .15y$

The constraints are:

$x \geq 0$

$y \geq 0$

The amounts of food must be non-negative.

The table gives a summary of nutrients provided:

| Food | Amount (in Ounces) | Total Consumption (in Grams) | | |
|---|---|---|---|---|
| | | Carbohydrates | Protein | Fats |
| A | $x$ | $10x$ | $2x$ | $3x$ |
| B | $y$ | $5y$ | $5y$ | $4y$ |
| Total | | $10x + 5y$ | $2x + 5y$ | $3x + 4y$ |

Daily requirements:

$10x + 5y \geq 200$

$2x + 5y \geq 100$

$3x + 4y \geq 120$

The LP model is:

Minimize: $C = .29x + .15y$

Subject to:

$10x + 5y \geq 200$          [Carbohydrates]

$2x + 5y \geq 100$          [Protein]

$3x + 4y \geq 120$          [Fats]

$x \geq 0$          [Non-negativity]

$y \geq 0$          [Non-negativity]

**EXAMPLE 4 (Investment Problem)**

Big Bros. Inc. is an investment company doing an analysis of the pension fund for a certain company. A maximum of #10 million is available to invest in two places. No more than #8 million can be invested in stocks yielding 12% and at least #2 million can be invested in long-term bonds yielding 8% . The stock-to-bond investment ratio cannot be more than 1 to 3 . How should Big Bros. advise

their client so that the pension fund will receive the maximum yearly return on investment?

Required: formulate the LP model.

## SOLUTION

To build this model, you need the simple interest formula:

$I = prt$ ; where,

$I =$ Interest: The amount paid for the use of another's money.

$p =$ Principal: The amount invested.

$r =$ Interest rate: Write this as a decimal. It is assumed to be an annual interest rate, unless otherwise stated.

$t =$ Time: In years, unless otherwise stated.

Let, $x =$ Amount invested in stocks [12% yield]

$y =$ Amount invested in bonds [8% yield]

Stocks: $I = .12x$, $p = x$, $r = .12$ and $t = 1$

Bonds: $I = .08y$, $p = y$, $r = .08$ and $t = 1$

Return on investment, R, is found by:

$R = .12x + .08y$ .

Constraints:

$x \geq 0$         [Investments are non-negativity]

$y \geq 0$         [Investments are non-negativity]

$x + y \leq 10$     [Maximum investment]

$x \leq 8$         [Not more than #8 million in stocks]

$y \geq 2$         [Not less than #2 million in bonds]

$3x \leq y$        [Stock to bond ratio of 1 to 3 i.e. must invest #3 million in bonds for every #1 million invested in stocks].

Thus, the LP model for this problem is::

Maximize: $R = .12x + .08y$

Subject to:

$$x + y \leq 10$$

$$x \leq 8$$

$$y \geq 2$$

$$3x - y \leq 0$$

$$x \geq 0$$

$$y \geq 0$$

## EXAMPLE 5 (Transportation Problem)

TRAX Ships a certain refrigerator unit from factories in Lagos and Onitsha to distribution centres in Abuja and Kano. Shipping costs are summarized in the table below:

| Source | Destination | Shipping cost |
|--------|-------------|---------------|
| Lagos | Abuja | #30 |
| | Kano | #40 |
| Onitsha | Abuja | #60 |
| | Kano | #50 |

The supply and demand, in number of units, is shown below:

| Supply | Demand |
|--------|--------|
| Lagos,  200 | Abuja,  300 |
| Onitsha,  600 | Kano,  400 |

How should shipments be made from Lagos and Onitsha to minimize the shipping cost?

**SOLUTION**

The information of this problem can be summarized by the following "map".
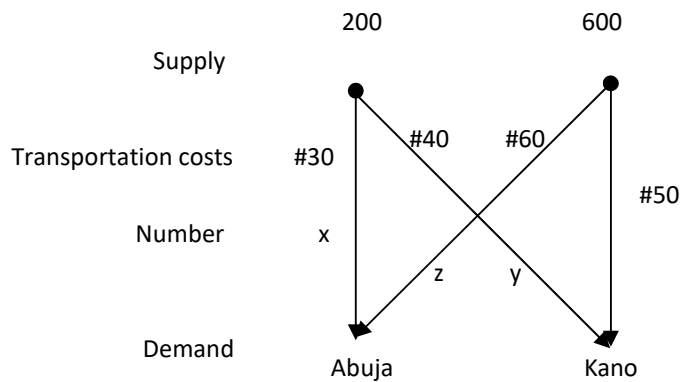


Suppose the following number of units is shipped.

| Source | Destination | Number | Shipping cost |
|--------|-------------|--------|---------------|
| Lagos | Abuja | $x$ | $30x$ |
|  | Kano | $y$ | $40y$ |
| Onitsha | Abuja | $z$ | $60z$ |
|  | Kano | $w$ | $50w$ |

The LP model is as follows:

Minimize: $C = 30x + 40y + 60z + 50w$

Subject to:

   $x + y \leq 200$          [Supply]

   $z + w \leq 600$          [Supply]

   $x + z \geq 300$          [Demand]

   $y + w \geq 400$          [Demand]

   $x \geq 0$, $y \geq 0$, $z \geq 0$, $w \geq 0$          [Non-negativity]

## 3.6  Graphical Solution of Linear Programming Problems

In the previous section, we looked at some models called linear programming models. In each case, the model had a function called an objective function, which was to be maximized or minimized while satisfying several conditions or constraints. If there are only two variables, we will use a graphical method of solution. We begin with the set of constraints and consider them as a system of inequalities. The solution of this system of inequalities is a set of points, S. Each point of the set S is called a feasible solution. The objective function can be evaluated for different feasible solutions and the maximum or minimum values obtained.

**EXAMPLE 6**

Maximize: $R = 4x + 5y$

Subject to:

   $2x + 5y \leq 25$

   $6x + 5y \leq 45$

$x \geq 0$, $y \geq 0$

## SOLUTION

To solve the above linear programming model using the graphical method, we shall turn each constraints inequality to equation and set each variable equal to zero $(0)$ to obtain two $(2)$ coordinate points for each equation (i.e. using double intercept form).

Having obtained all the coordinate points, we shall determine the range of our variables which enables us to know the appropriate scale to use for our graph. Thereafter, we shall draw the graph and join all the coordinate points with required straight line.

$2x + 5y = 25$ [Constraint 1]

When $x = 0$, $y = 5$ and when $y = 0$, $x = 12.5$.

$6x + 5y = 45$ [Constraint 2]

When $x = 0$, $y = 9$ and when $y = 0$, $x = 7.5$.

Minimum value of $x$ is $x = 0$.

Maximum value of $x$ is $x = 12.5$.

Range of $x$ is $0 \leq x \leq 12.5$.

Minimum value of $y$ is $y = 0$.

Maximum value of $y$ is $y = 9$.

Range of $y$ is $0 \leq y \leq 9$.
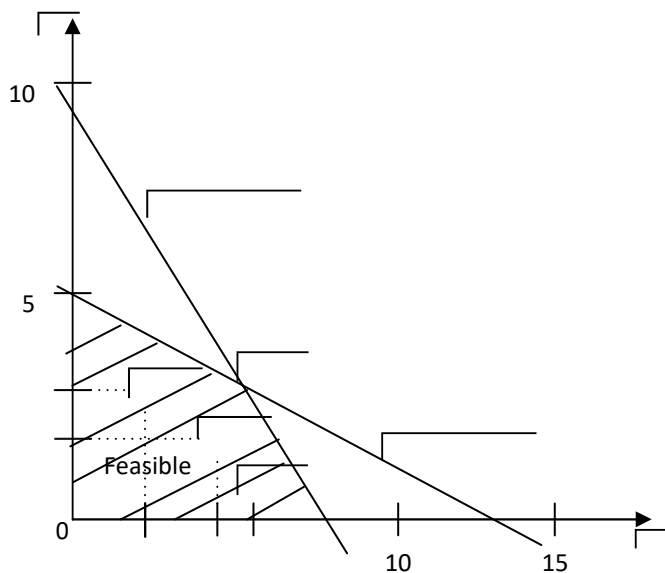


Fig. 1

The constraints give a set of feasible solutions as graphed above. To solve the linear programming problem, we must now find the feasible solution that makes the objective function as large as possible. Some possible solutions are listed below:

| Feasible solution (A point in the solution set of the system) | Objective function $R = 4x + 5y$ |
|---|---|
| (2,3) | $4(2) + 5(3) = 8 + 15 = 23$ |
| (4,2) | $4(4) + 5(2) = 16 + 10 = 26$ |
| (5,1) | $4(5) + 5(1) = 20 + 5 = 25$ |
| (7,0) | $4(7) + 5(0) = 28 + 0 = 28$ |
| (0,5) | $4(0) + 5(5) = 0 + 25 = 25$ |

In this list, the point that makes the objective function the largest is $(7,0)$. But, is this the largest for all feasible solutions? How about $(6,1)$? or $(5,3)$? It turns out that $(5,3)$ provide the maximum value: $4(5) + 5(3) = 20 + 15 = 35$.

**EXAMPLE 7**

Find the corner points for:

$2x + 5y \leq 25$

$6x + 5y \leq 45$

$x \geq 0$, $y \geq 0$

This is the set of feasible solution for Example 6 .

**SOLUTION**

The graph for Example 6 is repeated here and shows the corner points.
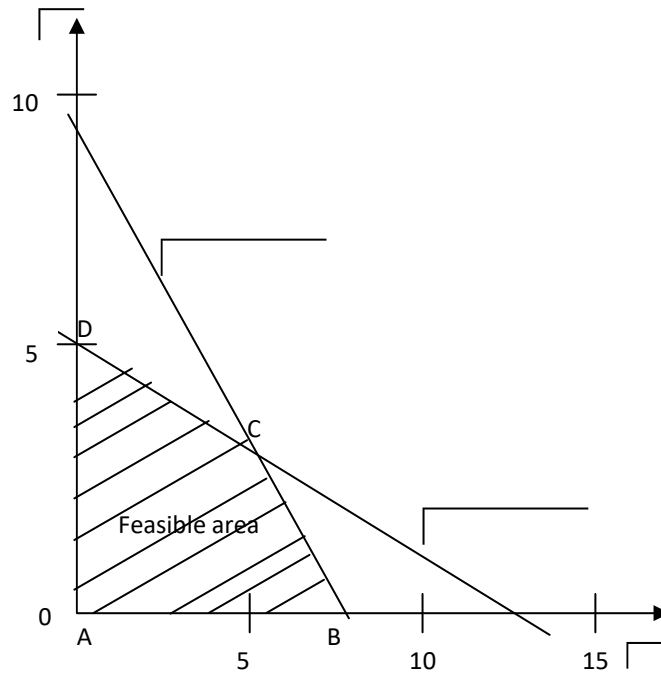


Fig. 5

Some corner points can usually be found by inspection. In this case, we can see $A = (0,0)$ and $D = (0,5)$. Some corner points may require some work with boundary lines (uses equations of boundaries not the inequalities giving the regions).

Point C:

System: $2x + 5y = 25$ … (1)

$\qquad 6x + 5y = 45$ … (2)

$\qquad (1) - (2) \Rightarrow -4x = -20$

$\qquad \Rightarrow x = 5$.

If $x = 5$, then from (1) or (2):

$\qquad y = 3$.

Point B:

System: $y = 0$  … (1)

$\qquad 6x + 5y = 45$ … (2)

$\qquad$ Solve by substitution:

$\qquad \Rightarrow 6x + 5(0) = 45$

$\qquad \Rightarrow x = \dfrac{45}{6} = 7.5$ .

The corner points for example 7 are: $(0,0)$ , $(0,5)$ , $(7.5,0)$ and $(5,3)$ .

Convex sets and corner points lead us to a method for solving certain linear programming problems.

## 3.7 <u>Linear Programming Theorem</u>

A linear expression in the variables $c_1 x + c_2 y$ defined over a convex set S whose sides are line segments, takes on its maximum value at a corner point of S and its minimum value at a corner point of S. If S is unbounded, there may or may not be an optimum value, but if there is, then it must occur at a corner point. In summary, to solve a linear programming problem graphically; the following steps must be taken:

- Find the objective expression (the quantity to be maximized or minimized).
- Find and graph the constraints defined by a system of linear inequalities; the simultaneous solution is called the set $S$ .
- Find the corner points of $S$ ; this may require the solution of a system of two equations with two unknowns, one of each corner points.
- Find the value of the objective expression for the coordinates of each corner point. The largest value is the maximum; the smallest value is the minimum.

### EXAMPLE 8

Solve graphically Example1.

### SOLUTION

The linear programming model is:

Maximize: $P = 143x + 60y$

Subject to:

$$x + y \leq 100$$

$$120x + 210y \leq 15000$$

$$110x + 30y \leq 4000$$

$$x \geq 0, \, y \geq 0.$$

Where $x$ the number of acres is planted in corn and $y$ is the number of acres planted in wheat.

First, graph the set of feasible solutions by graphing the system of inequalities, as shown in the figure:

$x + y = 100$ [Constraint 1]

When $x = 0$, $y = 100$ and when $y = 0$, $x = 100$.

$120x + 210y = 15000$ [Constraint 2]

When $x = 0$, $y = \dfrac{500}{7}$ and when $y = 0$, $x = 125$.

$110x + 30y = 4000$ [Constraint 3]

When $x = 0$, $y = \dfrac{400}{3}$ and when $y = 0$, $x = \dfrac{400}{11}$.



Next, find the corner points. By inspection $A = (0,0)$

Fig.

53
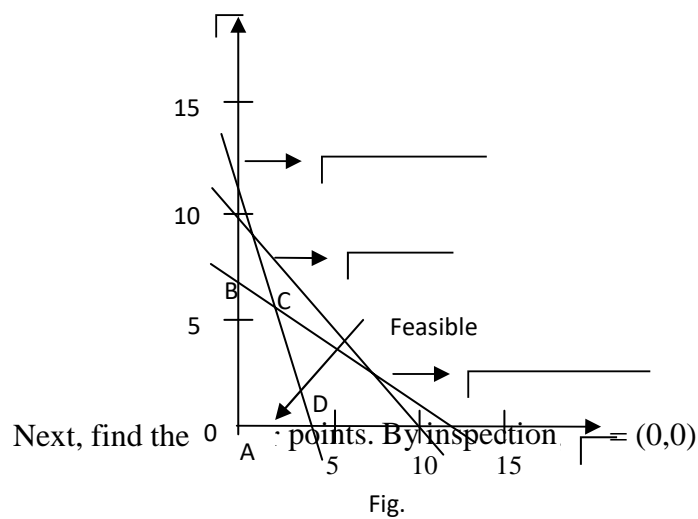
Point B:

System: $120x + 210y = 15000$ ... (1)

$x = 0$ ... (2)

Solve (1) and (2) simultaneously by substituting for $x = 0$ in (1):

$\Rightarrow 120(0) + 210y = 15000$

$\Rightarrow y = \dfrac{15000}{210} = \dfrac{500}{7}$.

Point B: $\left(0, \dfrac{500}{7}\right)$.

Point C:

System: $110x + 30y = 4000$ ... (1)

$120x + 210y = 15000$ ... (2)

$7(1) - (2) \Rightarrow 650x = 13000$

$\Rightarrow x = 20$.

Substitute for $x = 20$ in (1):

$\Rightarrow 110(20) + 30y = 4000$

$\Rightarrow 30y = 1800$.

$\therefore y = 60$

Point C: $(20,60)$.

Point D:

System: $110x + 30y = 4000$ ... (1)

$y = 0$ ... (2)

Solve (1) and (2) simultaneously by substituting for $y = 0$ in (1):

$\Rightarrow 110x + 30(0) = 4000$.

54

$$\Rightarrow 110x = 4000$$

$$\therefore x = \frac{400}{11}$$

Point D: $\left(\dfrac{400}{11}, 0\right)$.

Use the linear programming theorem and check the corner points:

| Corner point | Objective function $P = 143x + 60y$ |
|---|---|
| (0,0) | $143(0) + 60(0) = 0$ |
| $\left(0, \dfrac{500}{7}\right)$ | $143(0) + 60(500/7) = 4{,}286$ |
| $\left(\dfrac{400}{11}, 0\right)$ | $143(400/11) + 60(0) = 5{,}200$ |
| $(20,60)$ | $143(20) + 60(60) = 6{,}460$ |

The maximum value of P is 6,460 at $(20,60)$. This means that to maximum profit, the farmer should plant 20 acres in corn, plant 60 acres in wheat and leave 20 acres unplanted.

Notice from the graph in Example 8 that some of the constraints could be eliminated from the problem and everything else would remain unchanged. For example, the boundary $x + y = 100$ was not necessary in finding the maximum value of P. Such a condition is said to be a superfluous constraint. It is not uncommon to have superfluous constraints in a linear programming problem. Suppose, however, that the farmer in Example 1 contracted to have the grain stored at neighboring farm and now the contract calls for at least 4,000 bushels to be stored. This change from $110x + 30y \leq 4000$ to $110x + 30y \geq 4000$, now makes the condition $x + y \leq 100$ important to the solution of the problem. Therefore, you must be careful about superfluous constraints even though they do not affect the solution at the present time.

**EXAMPLE 9**

Solve the following linear programming problem:

Minimize: $C = 60x + 30y$

Subject to:

$$2x + 3y \geq 120$$

$$2x + y \geq 80$$

$$x \geq 0 , \ y \geq 0 .$$

**SOLUTION**
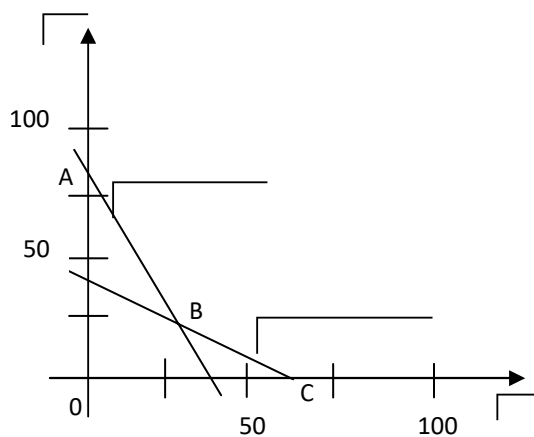


Fig. 7

Corner points $A = (0,80)$ and $C = (60,0)$ are found by inspection.

Point B:

System: $2x + 3y = 120$ … (1)

$2x + y = 80$ … (2)

$(1) - (2) \Rightarrow 2y = 40$

$\Rightarrow y = 20 .$

Substitute for $y = 20$ in (2) :

$\Rightarrow 2x + 20 = 80$.

$\Rightarrow 2x = 60$.

$\Rightarrow x = 30$.

Point B: $(30,20)$.

**Extreme values**.

| Corner point | Objective function $C = 60x + 30y$ |
|:---:|:---:|
| (0,80) | $60(0) + 30(80) = 2400$ |
| $(30,20)$ | $60(30) + 30(20) = 2400$ |
| $(60,0)$ | $60(60) + 30(0) = 3600$ |

From the table above, there are two minimum values for the objective function: $A = (0,80)$ and $B = (30,20)$. In this situation, the objective function will have the same minimum value (2,400) at all points along the boundary line segment A and B.

## Special Cases

Linear programming problems do not always yield a unique optimal solution. There are a number of special cases and we shall consider just two of them:

i.  No feasible solution and;

ii. Multiple optimum solutions.

No Feasible Solution

If the constraints are mutually exclusive, no feasible area can be defined and no optimum solution can exist. Consider again the maximization problem.

**EXAMPLE 10**

Maximize: $z = 2x + 3y$

Subject to:

$$x + 2y \leq 40$$

$$6x + 5y \leq 150$$

$$x \geq 0,\ y \geq 0$$

**SOLUTION**

The feasible area is defined by the constraints as shown in the figure below.
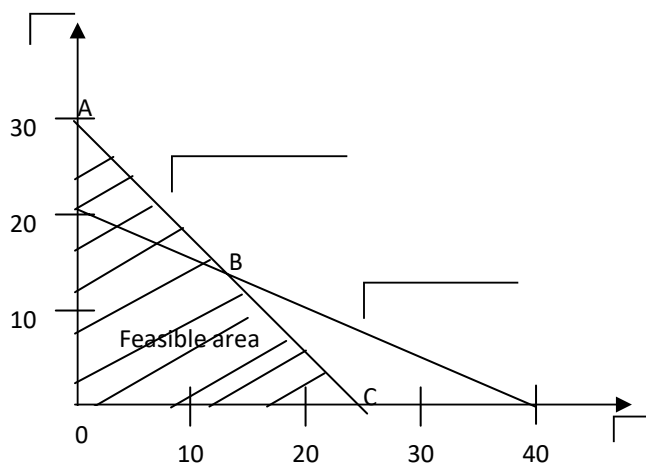


Fig. 8

Suppose that in addition to the existing constraints, the company is contracted to produce at least 30 units each week. This additional constraint can be written as: $x + y \geq 30$. As a boundary solution, the constraint would be: $x + y = 30$, $(x = 0, y = 30)(x = 30, y = 0)$.

The three structural constraints are shown in the figure below.

58

This case presents the manager with demands which cannot simultaneously be satisfied.
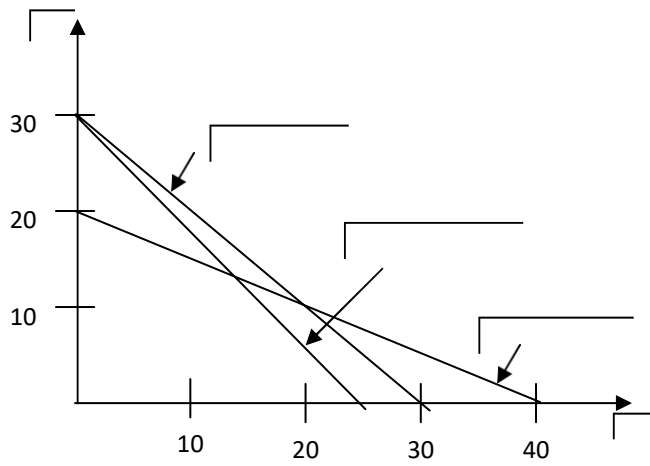


Fig. 9

Multiple Optimum Solutions

A multiple optimum solution results when the objective function is parallel to one of the boundary constraints. Consider the following problem.

**EXAMPLE 11**

Minimize: $z = 600x + 900y$

Subject to:

$$40x + 60y \geq 480$$

$$30x + 15y \geq 180$$

$$x \geq 0, \, y \geq 0$$

**SOLUTION**

If we let $z = \#8100$, then:

$8100 = 600x + 900y$, $(x = 0, \, y = 9)(x = 13.5, \, y = 0)$.

The resultant trial cost is shown in the figure below.
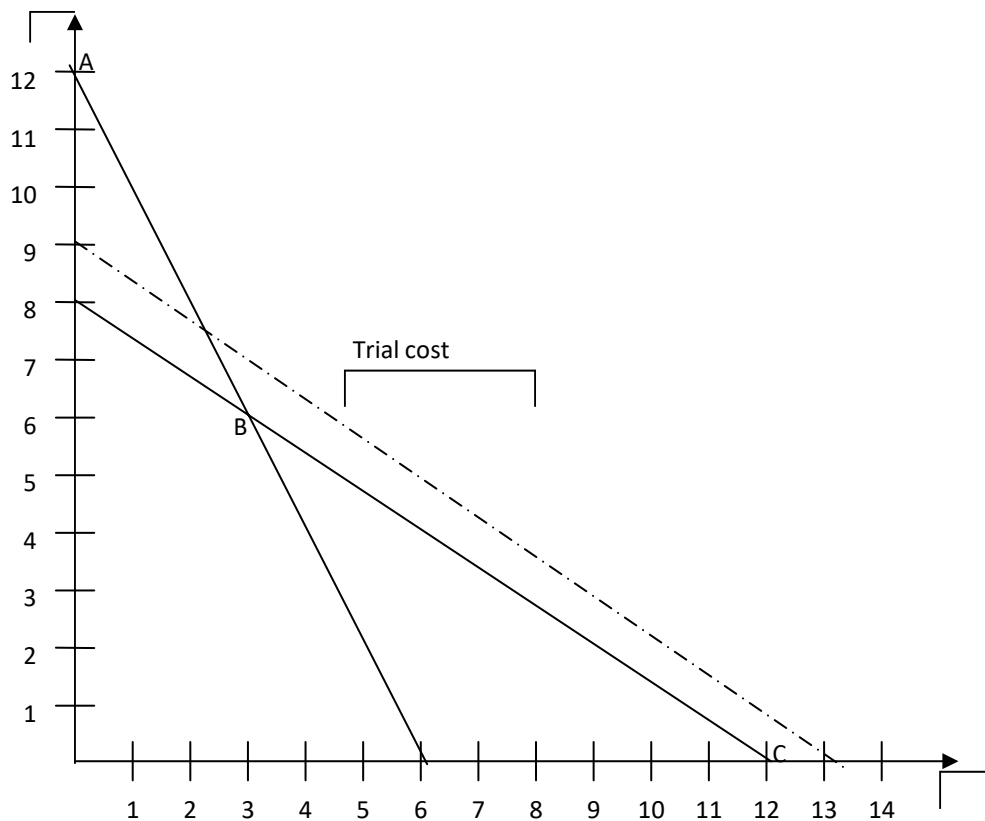


Fig. 10

This line is parallel to the boundary line BC. The lowest acceptable cost solution will be coincidental with the line BC making point B, point C and any other points on the line BC optimal. Multiple optimum solutions present the manager with choice and hence some flexibility.

## **The value of Resources**

Linear programming provides a method for evaluating the marginal value of resources. Consider yet again the maximization problem.

## **EXAMPLE 12**

Maximize: $z = 2x + 3y$

Subject to:

$x + 2y \leq 40$ (Labour hours)

$6x + 5y \leq 150$ (Moulding material)
$x \geq 0$, $y \geq 0$

In this case, the solution is limited by the 40 hours of labour and the 150 litres of moulding material. To assess the value of additional resources, we can consider what difference it would make if we could provide an extra hour of labour or an extra unit of moulding material. The amount added to profit in this case (or more generally, $z$ in the objective function) as a result of the additional unit of resources is seen as the marginal value of the resources and is referred to as the "opportunity cost" or the "shadow price".

To determine the shadow price of labour, we would increase the hours available from 40 to 41. The linear programming formulation now becomes:

Maximize: $z = 2x + 3y$

Subject to:

$$x + 2y \leq 41$$

$$6x + 5y \leq 150$$

$$x \geq 0, \ y \geq 0$$

This type of marginal analysis is difficult to show graphically because of the small movements involved. Effectively, the labour constraint has moved outwards and can be plotted using the points $(x = 0, y = 20.5)$ and $(x = 41, y = 0)$.

The new solution is $x = 13\frac{4}{7}$ and $y = 13\frac{5}{7}$.

The new level of profit can be found by substitution into the objective function, thus:

$$z = \left(\#2 \times 13\frac{4}{7}\right) + \left(\#3 \times 13\frac{5}{7}\right) = \#68.29.$$

The increase in profit resulting from the additional hour of labour, or shadow price of labour, is the difference between the new profit and the old profit i.e. $\#(68.29 - 67.14) = \#1.15$.

To determine the shadow price of materials, we would increase the number of litres available from 150 to 151. The linear programming formulation now becomes:

Maximize: $z = 2x + 3y$

Subject to:

$$x + 2y \le 40$$

$$6x + 5y \le 151$$

$$x \ge 0, y \ge 0$$

In this case, it is the material constraint that would move outwards while the labour constraint remained unchanged at 40 hours. To plot the new material constraint, the points $(x = 0, y = 30.2)$ and $(x = 25.17, y = 0)$ may be used.

The new solution is $x = 14\frac{4}{7}$ and $y = 12\frac{5}{7}$.

The new level of profit can be found by substitution into the objective function, thus:

$$z = \left( \#2 \times 14\frac{4}{7} \right) + \left( \#3 \times 12\frac{5}{7} \right) = \#67.29.$$

The increase in profit resulting from the additional litre of moulding material, or shadow price of material, is the difference between the new profit and the old profit i.e.
$\#(67.29 - 67.14) = \#0.15$.

If the manager were to pay below $\#1.15$ for the additional hour of labour (unlikely to be available at these rates!), then profits could be increased, and if the manager were to pay above this figure, then profits would decrease. Similarly, if the manager can pay below $\#0.15$ for an additional unit of moulding material, then profits can be increased but if the manager were to pay above this

level, then profits would decrease. It is useful to see the effect of increasing both labour and materials by one unit.

The linear programming formulation now becomes:

Maximize: $z = 2x + 3y$

Subject to:

$$x + 2y \leq 41$$

$$6x + 5y \leq 151$$

$$x \geq 0, y \geq 0$$

The new solution is $x = 13\frac{6}{7}$ and $y = 13\frac{4}{7}$.

The new level of profit is $z = \left(\#2 \times 13\frac{6}{7}\right) + \left(\#3 \times 13\frac{4}{7}\right) = \#68.43$.

The increase in profit is $\#(68.43 - 67.14) = \#1.29$.

This increased profit (subject to the small rounding error of $\#0.01$) is the sum of the shadow prices $\#(1.15 + 0.15)$. It should be noted that, the shadow prices calculated only apply while the constraints continue to work in the same way. If, for example, we continue to increase the supply of moulding material (because it can obtained at a market price below the shadow price), other constraints may become active and change the value of the shadow price.

## 4.0. CONLUSION

Linear programming model have four properties, viz: Objective function, Alternative decision variables, Constraint and linear representations of the objective function and the constraints

## 5.0. SUMMARY

* Linear programming is an aspect of operations research that utilizes the construction of a mathematical model to solve allocation problem.

. * In formulating a mathematical model we follow these steps:

(i) Identification of decision variables

(ii) Statement of the objective function

(iii) Statement of the constraint

* Graphical method can only be use to solve a linear programming model involving two decision variables.

## 6.0. TUTOR MARKED ASSIGNMENT

1. In relation to Linear Programming, explain the implication of the following assumptions of the model:

(i) Linearity of the objective function and constraints

(ii) Certainty and (iii) Decision variables

2. Orient Paper Mills produces two grades of paper X and Y Because of raw material restrictions not more than 400 tons of grade X and not more than 300 tons of grade Y can be produced in a week. There are 160 production hour in a week. It requires 0.2 and 0.4 hour to produced a ton of products X and Y respectively. With corresponding profit of #20 and # 50 per ton.

Required:

(a) Formulate a linear programming model to optimize the product mix for maximum profit.

(b) Solve graphical the model in (a) above.

## 7.0. REFRENCES/FURTHER READINGS

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## UNIT 3        LINEAR PROGRAMMING (SIMPLEX METHOD)

1.0.    Introduction

2.0.    Objective

3.0.    Main Content

3.1. Various steps in solving or problems using simplex method.

(a)       Maximization problems

(b)       Minimization problems

## 1.0    INTRODUCTION

We have seen in the chapter on Linear Programming Problems that one can conveniently solve problems with two variables. If we have more than two variables, the solution becomes very cumbersome and complicated. Thus, there is a limitation of LPP.

Simplex method is an algebraic procedure in which a series of repetitive operation are use\d and we progressively approach the optimal solution. Thus, this procedure has a number of steps to find the solution to any problems consisting of any number of variables and constraints, however problems with more than 4 variables cannot be solved manually and require the use of computer for solving them.

This method developed by the American mathematician G B Dantizg, can be used to solve any problem, which has a solution. The process of reaching the optimal solution through different stages is also called iterative, because4 the same computational steps are repeated a number of times before the optimum

## 2.0    OBJECTIVE

  ✓  Preparation of LPD for use of simplex

  ✓  Understanding of the need of use of simplex

  ✓  Steps involve in using a simple method.

- ✓ Preparing a simplex Table and understanding its various components.

- ✓ Demonstrating the use of simplex method for solving a LLP

- ✓ Solving LPP using maximization problem

## 3.0 MAIN TEXT

### 3.1. VARIOUS STEPS IN SOLVING OR PROBLEM USING SIMPLE METHOD.

Step 1.        **Formulate the problem**

The problem must be put in the form of a mathematical model. The standard form of the LP model has the following proprieties

     a      An objective function, which has to maximized or minimized

     b      All the constraints can be put in the form equations.

     c      All the variables are non-negative.

**Step II.        Set up the initial simple table with slack variable or surplus variables in the solution.**

A constraint of type $\leq$ or $\geq$ can be converted into an equation by adding a slack variable or subtracting a surplus variable on the left hand side of the constraint.

For example, in the constraint $X_1 + 3X_2 \leq 15$ we add a slack $S_1 \geq 0$ To The left side to obtain an equation : $X_1 + 3X_2 + S_1 = 15$, $S_1 \geq 0$

Now consider the constraint $2X_1 + 3X_2 - X_3 = \geq 4$, since the left side is not smaller than the right side we subtract a surplus variable $S_2 {}_{> 0}$ from the left side to obtain the equation

$$2X_1 + 3X_2 - X_3 - S_2 = 4 , S_2 \geq 0$$

The use of the slack variable or surplus variable will become clear in the actual example as we proceed.

**Step III.** Determine the decision variables which are to be brought in the solution

**Step IV.** Determine which variable to replace.

**Step V.** Calculation new row values for entering variables

**Step VI.** Revise remaining rows.

Repeat steps III to VI till an optimal solution is obtained. This procedure can best be explained with the help of a suitable example.

**Example 1** Solve the following linear programming problem by simplex method.

Maximize      $Z = 10X_1 + 20X_2$

Subject to the following constraints

$$3X_1 + 2X_2 \leq 1200$$

$$2X_1 + 6X_2 \leq 1500$$

$$X_1 \leq 350$$

$$X_2 \leq 200$$

$$X_1, X_2 > 0$$

**Solution**

**Step 1.** Formulate the problem

Problem is already stated in the mathematical model.

**Step 2.** Set up the initial simplex table with the slack variables in solution. By introducing the slack variables, the equation in step I, *i.e.* the mathematical model can be rewritten as follows

$$3X_1 + 2X_2 + S_1 = 1,200$$

$$2X_1 + 6X_2 + S_2 = 1,500$$

$$X_1 + S_{3\text{-}} \quad = 350$$

$$X_2 + S_4 = 200$$

Where $S_1$ $S_2$ $S_3$ and $S_4$ are the slack variables. Let us re- write the above equation in symmetrical manner so that all the four slacks $S_1$, $S_2$, $S_3$ and $S_4$ appear in all equation

$$3X_1 + 2X_2 + 1S_1 + 0S_2 + 0S_3 + 0S_4 = 1,200$$

$$2X_1 + 6X_2 + 0S_1 + 1S_2 + 0S_3 + 0S_4 = 1,500$$

$$1X_1 + 0X_2 + 0S_1 + 0S_2 + 1S_3 + 0S_4 = 350$$

$$0X_1 + 1X_2 + 0S_1 + 0S_2 + 1S_3 + 0S_4 = 200$$

Let us also write the objective function Z by introducing the slack in it.

$$Z = 10X_1 + 20X_2 + 0S_1 + 0S_2 + 1S_3 + 0S_4$$

The first simplex table can now be written as.

**Table 1**

| $C_1$ | Solution Mix | # 10 | # 20 | 0 | 0 | 0 | 0 | Contribution unit quantity |
|---|---|---|---|---|---|---|---|---|
| | | $X_1$ | $X_1$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | |
| 0 | $S_1$ | 3 | 2 | 1 | 0 | 0 | 0 | 1200 |
| 0 | $S_2$ | 2 | 6 | 0 | 1 | 0 | 0 | 1500 |
| 0 | $S_3$ | 1 | 0 | 0 | 0 | 1 | 0 | 350 |
| 0 | $S_4$ | 0 | (1) Key Element | 0 | 0 | 0 | 1 | 200 Key Row ← |
| $Z_j$ | | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $(C_J - Z_J)$ | | 10 | 20 ↑ | 0 | 0 | 0 | 0 | |

| | | Key Column | | | | |
|---|---|---|---|---|---|---|
| | | | | | | |

The first simplex table is shown in Table 1. The table is explained as below.

1. Row 1 contains C or the contribution to total profit with the production of one unit of each product $X_1$ and $X_2$. This row gives the coefficients of the variables in the objective function which will remain the same. Under column $1(C_J)$ is provided profit unit of 4 variables $S_1$ $S_2$ $S_3$ $S_4$ which is zero.
2. All the variables $S_1$, $S_2$, $S_3$, $S_4$ are listed under solution Mix. Their profit is zero and written under column 1 $(C_J)$ as explained above.
3. The constraints variables are written to the right of solution mix. These are $X_1$ $X_2$ $S_1$, $S_2$, $S_3$ and $S_4$ Under these are written coefficient of variable and under each are written the coefficients $X_1$, $X_2$, $S_1$, $S_2$, $S_3$ and $S_4$ in first constraint equation are 3,2,1,0,0 and 0, respectively which are written under these variables in the first level. Similarly,, the remaining 3 rows represents the coefficient of the variables as they appear in the other 3 constraint equation The entries in the quantity column represent the right hand side of each constraint equation. These values are 1,200, 1,500, 350 and 200 receptivity, for the given problem.
4. The $Z_j$ values in the second row form the bottom refer to the amount of gross profit that is given up by the introducing one unit the solution. The subscript $j$ refer to the specific variable being considered. The $Z_j$ values under the quantity column is the total profit for their solution. In the initial column all the $Z_j$ - values will be zero because no real product is being manufactured and hence there is on gross profit to be lost if they are replaced.
5. The bottom row of the table contains net profit per unit obtained by introducing one unit of a given variable into the solution. This row is designated as the $C_J – Z_j$ row. The procedure for calculating $Z_j$ and $C_j – Z_j$ values is given below.

**Calculation of $Z_J$**

| $C_j$ x $X_1$ | $C_j$ x $X_2$ | $C_j$ x $S_1$ |
|---|---|---|
| 0 x 3 = 0 | 0 x 2 = 0 | 0 x 1 = 0 |

$$+ \qquad\qquad +$$

$$+$$

| $0 \times 2 = 0$ | $0 \times 6 = 0$ | $0 \times 0 = 0$ |
|:---:|:---:|:---:|
| $+$ | $+$ | $+$ |
| $0 \times 1 = 0$ | $0 \times 0 = 0$ | $0 \times 0 = 0$ |
| $+$ | $+$ | $+$ |
| $\dfrac{0 \, x \, 0 = 0}{Zx_1 = 0}$ | $\dfrac{0 \, x \, 1 = 0}{Zx_2 = 0}$ | $\dfrac{0 \, x \, 0 = 0}{Zs_1 = 0}$ |

Similarly, $Zs_2$, $Zs_2$ and $Zs_4$ can be calculated as 0 each

**Calculation of $C_j$ - $Z_j$**

$$Cx_1 - Zx_1 = 10 - 0 = 10$$

$$Cx_2 - Zx_2 = 20 - 0 = 20$$

$$Cs_1 - Zs_1 = 0 - 0 \quad = 0$$

$$Cs_2 - Zs_2 = 0 - 0 \quad = 0$$

$$Cs_3 - Zs_3 = 0 - 0 \quad = 0$$

$$Cs_4 - Zs_4 = 0 - 0 \quad = 0$$

The total profit for this solution is # zero.

**Step 3.**

Determine the variable to be brought into the solution. An improved solution is possible if there is a positive value in $C_j - Z_j$ row. The variable with the largest positive value in the $C_j - Z_j$, row is subjected as the objective is to maximize the profit, The column associated with this variable is referred to as 'Key column' and is designated by a small arrow beneath this column. In the given example , 20 is the largest possible value corresponding to $X_2$ which is selected as the key column.

**Step 4.**

Determine which variable is to be replaced. To make this determination, divine each amount in the contribution quantity column by the amount in the comparable row of Key column, $X_2$ and choose the variable associated with smallest quotient as the one to be replace. In the given example, these value are calculated as

for the $S_1$ row – $1200/2 = 600$

for the $S_2$ row – $1500/6 = 250$

for the $S_3$ row – $350/2 = \infty$

for the $S_4$ row – $200/1 = 200$

Since the smallest quotient is 200 corresponding to $S_4$, $S_4$ will be replaced, and its row is identified by the small arrow to the right of the table as shown. The quotient represents the maximum value of X which could be brought into the solution.

**Step 5.**

Calculate the new row values for entering the variable. The introduction of $X_2$ into the solution requires that the entire $S_4$ row be replaced. The values of $X_2$, the replacing row, are obtained by dividing each value presently in the $S_4$ row by the value in column $X_2$ in the same row. This value is termed as the key or the pivotal element since it occurs at the intersection of key row and key column.

$\longleftarrow X_2 \longrightarrow$          key column

2

6        key element

0

$S_4$    0    1    0    0    0    1      200 key row

20

The row values entering variable $X_2$ can be calculated as follows:

$0/1 = 0$ ; $1/1 = 1$ ; $0/1 = 0$ ; $0/1 = 0$ ; $0/1 = 0$; - $1/1 = 1$; $200/1 = 200$

**Step 6.**

Update the remaining rows. The new $S_2$ row values are 0, 1, 0, 0, 1 and 200 which are same as the previous table as the key element happens to be 1. The introduction of a new variable into the problem will affect the values of remaining variables and a second set of calculations need to be performed to update the initial table. These calculations are performed as given here:

Updated $S_1$ row = old $S_1$ row – intersectional element of old $S_1$ row x corresponding element of new $X_2$ row.

$$= 3 - [2 \times 0] = 3$$

$$= 2 - [2 \times 1] = 0$$

$$= 1 - [2 \times 0] = 1$$

$$= 0 - [2 \times 0] = 0$$

$$= 0 - [2 \times 0] = 0$$

$$= 0 - [2 \times 1] = -2$$

$$= 1200 - [2 \times 200] = 800$$

Similarly, the updated elements of $S_2$ and $S_3$ rows can be calculated as follows:

| **Elements of updated $S_2$ row** | **Elements of updated $S_3$ row** |
|---|---|
| $2 - [6 \times 0] = 2$ | $1 - [0 \times 0] = 1$ |
| $6 - [6 \times 1] = 0$ | $0 - [0 \times 1] = 0$ |
| $0 - [6 \times 0] = 0$ | $0 - [0 \times 0] = 0$ |
| $1 - [6 \times 0] = 1$ | $0 - [0 \times 0] = 0$ |

$$0 - [6 \times 0] = 0$$

$$0 - [6 \times 1] = -6$$

$$1500 - [6 \times 200] = 300$$

Rewiring the second simplex table with the updated elements as shown in Table 2. below

**Table 2**

|  | Solution | # 10 | # 20 | 0 | 0 | 0 | 0 | Contribution |  |
|---|---|---|---|---|---|---|---|---|---|
| $C_i$ | Mix | $X_1$ | $X_2$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | Quantity | Ratio |
| 0 | $S_1$ | 3 | 0 | 1 | 0 | 0 | -2 | 800 | 266.7 |
| 0 | $S_2$ | 2 | 0 | 0 | 1 | 0 | -6 | 300 | 150 → |
| 0 | $S_3$ | 1 | 0 | 0 | 0 | 1 | 0 | 350 | 350 |
| 20 | $X_2$ | 0 | 1 | 0 | 0 | 0 | 1 | 200 | ∞ |
|  | $Z_j$ | 0 | 20 | 0 | 0 | 0 | 20 | 400 |  |
|  | $(C_j - Z_j)$ | 10 | 0 | 0 | 0 | 0 | -20 |  |  |

The new variable entering the solution would be $X_1$. It will replace the $S_2$ row which can be shown as follows:

for the $S_1$ row – 800/2 = 266.7

for the $S_2$ row – 300/6 = 150

for the $S_3$ row – 350/2 = 350

for the $S_4$ row – 200/1 = ∞

As he quotient 150 corresponding of $S_2$ row is the minimum, it will be replaced by $X_1$ in the new solution. The corresponding elements of $S_2$ row can be calculated as follows:

$X_1$

3                         Key element

$S_2$    2    0    0    0    1    0    -6    300 Key row

1

0

0

10

Key column

New element of $S_2$ row to be replace by $X_1$ are

$2/2 = 1$ ; $0/2 = 0$ ; $0/2 = 0$ ; $1/2 = 1/2$ ; $0/2 = 0$; $- 6/2 = -3$;  $300/2 = 150$;

The updated elements of $S_1$ and $S_3$ rows can be calculated as follows:

| **Elements of updated $S_1$ row** | **Elements of updated $S_3$ row** |
|---|---|
| $3 – [3 \times 1] = 0$ | $1 – [1 \times 1] = 0$ |
| $0 – [3 \times 0] = 0$ | $0 – [1 \times 0] = 0$ |
| $1 – [3 \times 0] = 1$ | $0 – [1 \times 0] = 0$ |
| $1 – [3 \times 1/2] = -3/2$ | $0 – [1 \times 1/2] = 1/2$ |
| $0 – [3 \times 0] = 0$ | |
| $-2 – [3 \times 3] = -7$ | |
| $800 – [3 \times 150] = 350$ | |

**Elements of updated $X_2$ row**

$$0 – [0 \times 1] = 0$$

$$1 – [0 \times 0] = 1$$

$$0 – [0 \times 0] = 0$$

$$0 – [0 \times 1/2] = 0$$

$$0 – [0 \times 0] = 0$$

$$1 – [0 \times \text{-}3] = 1$$

$$200 – [0 \times 150] = 200$$

Revised simplex table can now be written as shown in Table 3 below:

| $C_i$ | Solution Mix | # 10 $X_1$ | # 20 $X_2$ | 0 $S_1$ | 0 $S_2$ | 0 $S_3$ | 0 $S_4$ | Contribution Quantity | Min Ratio |
|---|---|---|---|---|---|---|---|---|---|
| 0 | $S_1$ | 0 | 0 | 1 | -3/2 | 0 | 7 | 350 | 50 |
| 10 | $X_1$ | 1 | 0 | 0 | 1/2 | 0 | -3 | 150 | -50 |
| 0 | $S_3$ | 0 | 0 | 0 | -1/2 | 1 | 3 | 200 | 66.7 |
| 20 | $X_2$ | 0 | 1 | 0 | 0 | 0 | 1 | 200 | 200 |
| | $Z_j$ | 10 | 20 | 0 | 5 | 0 | -10 | 5500 | |
| | $(C_j – Z_j)$ | 0 | 0 | 0 | -5 | 0 | 10 | | |

Now the new entering variable will be $S_4$ and it will replace $S_1$ as shown below:

$$350/7 = 50$$

$$150/\text{-}3 = \text{-}50$$

$$200/3 = 66.7$$

$$200/1 = 200$$

In these figures, 50 represent the minimum quotient which corresponds to row $S_1$. The negative sign is not considered.

The new elements of $S_1$ row to be replaced by $S_4$ can be calculated as follows:

$S_4$

$S_1$   0        0        1      -3/2   0 ←——— 7      350                        key
row

                                                           -3

                                                            3

                                                            1

                                                           -10

                                                            10

                                   ↑
                                                           Key column

The new elements of $S_4$ row would be

$0/7 = 0$; $0/7 = 0$; $1/7 = 1/7$; $(-3/2) \times (1/7) = -3/14$; $0/7 = 0$; 1; $7/7 = 1$; $350/7 = 50$

The updated elements of the other rows can be calculated as follows:

**Elements of updated $X_1$ row**

$1 - [-3 \times 0] = 1$

$0 - [-3 \times 0] = 0$

$0 - [-3 \times 1/7] = 3/7$

$\dfrac{1}{2} - [-3 \times 3/14] = 1/7$

$0 - [-3 \times 0] = 0$

$-3 - [-3 \times 1] = 0$

$150 - [-3 \times 50] = 300$

**Elements of updated $S_3$ row**

$0 - [3 \times 0] = 0$

$0 - [3 \times 0] = 0$

$0 - [3 \times 1/7] = 3/7$

$-\dfrac{1}{2} - [3 \times 3/14] = -1/7$

**Elements of updated $X_2$ row**

$0 - [1 \times 0] = 0$

$1 - [1 \times 0] = 1$

$0 - [1 \times 1/7] = 1/7$

$0 - [1 \times -3/14] = 3/14$

$0 - [1 \times 0] = 0$

$1 - [1 \times 1] = 0$

$200 - [1 \times 50] = 150$

Revised simplex table can now be written as shown in Table 4 below:

|  | Solution | # 10 | # 20 | 0 | 0 | 0 | 0 | Contribution |
|---|---|---|---|---|---|---|---|---|
| $C_i$ | Mix | $X_1$ | $X_2$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | Quantity |
| 0 | $S_4$ | 0 | 0 | 1/7 | -3/14 | 0 | 1 | 50 |
| 10 | $X_1$ | 1 | 0 | 3/7 | -1/7 | 0 | 0 | 300 |
| 0 | $S_3$ | 0 | 0 | -3/7 | 1/7 | 1 | 0 | 50 |
| 20 | $X_2$ | 0 | 1 | -1/7 | 3/14 | 0 | 0 | 150 |
| | $Z_j$ | 10 | 20 | 10/7 | 40/14 | 0 | 0 | 6,000 |
| | $(C_j - Z_j)$ | 0 | 0 | -10/7 | -40/14 | 0 | 0 | |

As there is no positive value in $C_j - Z_j$ row it represents the optimal solution, which is given as:

$X_1 = 300$ units : $X_2 = 150$ units

And the maximum profit $Z = $ # 6,000

*Minimization problems.*

Identical procedure is followed for solving the minimization problems. Since the objective is to minimize rather than maximize, a negative ($C_j$ – $Zj$) value indicates potential improvement. Therefore, the variable associated with largest negative ($C_j$ – $Zj$) value would be brought into the solution first. Additional variables are brought in to set up such problems. However, such problems involve greater than or equal to constraints, which need to be treated separately from less than or equal to constraints, which are typical of maximization problems. In order to convert such inequalities, the following procedure may be adopted.

For example, if the constrain equation is represented as:

$$3X_1 + 2X_2 > 1200$$

To convert this into equality, it would be written as:

$$3X_1 + 2X_2 - S_1 = 1200$$

Where $S_1$ is a slack variable. However, this will create a difficulty in the simplex method because of the fact that the initial simplex solution starts with slack variables and a negative value ($-1S_1$) would be in the solution, a condition which is not permitted in linear programming. To overcome this problem, the simplex procedure requires that another variable known as artificial variable be added to each equation in which a slack variable is subtracted. An artificial variable may be thought of as representing a fictitious product having very high cost which though permitted in the initial solution to a simplex problem, would never appear in the solution. Defining A as an artificial variable, the constraint equation can be written as:

$$3X_1 + 2X_2 - 1S_1 + 1A_1 = 1200$$

Assuming the objective function is to minimize cost, it would be written as:

$$10X_1 + 20X_2 - 0S_1 + MA_1 \text{ to be minimized}$$

Where M is assumed to be very large cost (say, 1 million). Also $S_1$ is added to the objective function even though it is negative in constraint equation. An artificial variable is also added to constraint equations with equality sign, e.g. if the constraint equation is

$$3X_1 + 2X_2 = 1200$$

then, in simplex it would change to

$$3X_1 + 2X_2 + 1A_1 = 1200$$

to satisfy simplex requirement and would be reflected as MA in the objective function

## Example 2

ABC company manufactures and sells two products $P_1$ and $P_2$. Each unit of $P_1$ requires 2 hours of mahining and 1 hour of skilled labour. Each unit of $P_2$ requires 1 hour of machining and 2 hours of labour. The machine capacity is limited to 600 machine hours and skilled labour is limited to 650 man hours. Only 300 units of product $P_1$ can be sold in the market. You are require to

a.    Develop a suitable model to determine the optimal product mix

b.   Find out the optimal product mix and the maximum contribution. Unit contribution from product $P_1$ is # 8 and from product $P_2$ is # 12.

c.     Determine the incremental contribution / unit of machine-hours, per unit of labour and per unit of product $P_1$.

## Solution

**Step 1.**             **Formulation of LP model**

Let $X_1$ and $X_2$ be the number of units to be manufactured of the two products $P_1$ and $P_2$ respectively. We are required to find out the number of units of the two products to be manufactured to maximize contribution, i.e., profits when individual contribution of the two products are given. LP model can be formulated as follows:

$$\text{Maximize } Z = 8X_1 + 12X_2$$

Subject to conditions/constraints

$2X_1 + X_2 \leq 600$ (Machine time constraint)

$X_1 + 2X_2 \leq 650$ (Labour-time constraint)

$X_1 \leq 300$ (marketing constraint of product $P_1$)

**Step 2.**                    **Converting constraints into equations**

LP problem has to be written in a standard form, for which the inequalities of the constraints have to be converted into equations. For this purpose, we add a slack variable to each constraint equation. Slack is the unused or spare capacity for the constraints to which it is added. In less than ($\leq$) type of constraint, the slack variable denoted by S, is added to convert inequalities into equations. S is always a non-negative figure or 0. if S is negative, it may be seen that the capacity utilized will exceed the total capacity, which is absurd. The above inequalities of this problem can be rewritten by adding suitable slack variables and converted into equations as shown below.

$2X_1 + X_2 + S_1 = 600$

$X_1 + 2X_2 \, S_2 = 650$

$X_1 \, S_3 = 300$

$X_1, X_2, S_1, S_2, S_3 > 0$

Slack variables $S_1$, $S_2$ and $S_3$ contribute zero to the objective function since they represent only unused resources. Let us include these slack variables in the objective function. Then maximize: $Z = 8X_1 + 12X_2 + 0S_1 + 0S_2 + 0S_3$

**Step 3.**             **Set up the initial solution**

Let us recollect that the computational procedure in the simplex method is based on the following fundamental property.

"The optimal solution to a Linear Programming problem always occurs at one of three corner points of the feasible solution space".

It means that the corner points of the feasible solution region can provide the optimal solution. Let the search start with the origin which means nothing is produced at origin (0, 0) and the value of decision variable $X_1$ and $X_2$ is zero. In such a case, $S_1$ = 600, $S_2$ = 650, $S_3$ = 300 are the spare capacities as nothing (0) is being produced. In the solution at origin we have two variables $X_1$ and $X_2$ with zero value and three variables ($S_1$, $S_2$ and $S_3$) with specific value of 600, 650 and 300. The variables with non-zero values, i.e. $S_1$, $S_2$ and $S_3$ are called the **basic variables** where as the other variables with zero values i.e. $X_1$, $X_2$ and $X_3$ are called **non-basic variables.** It can be seen that the number of basic variables is the same as the number of constraints equations (three in the present problem). The solution with basic variables is called basic solution which can be further divided into Basic Feasible Solution and Basic Infeasible Solution. The first type of solutions are those which satisfy all the constraints. In Simplex Method, we sear for basic feasible solution only.

**Step 4.**             **Developing initial simplex table**

The initial decision can be put in the form of a table which is called a Simplex Table or Simplex Matrix. The details of the matrix are as follows

(1) Row 1 contains $C_j$ or the contribution to total profit with the production of one unit of each product $P_1$ and $P_2$. Under column 1 ($C_j$) are listed the profit coefficients of the basic variables. In the present problem, the profit coefficients of $S_1$, $S_2$ and $S_3$ are zero.
(2) In the column labeled Solution Mix or Product Mix are listed he variables $S_1$, $S_2$ and $S_3$, their profits are zero and written under column 1 ($C_j$) as explained above.
1. In the column labeled 'contribution unit quantity' are listed the values of basic variables included in the solution. We have seen in the initial solution $S_1$ = 600, $S_2$ = 650 and $S_3$ = 300. These values are listed under this column on the

right side as shown in table 3.5. Any variables not listed under the solution-mix column are the non-basic variables and their values are zero.

2. The total profit contribution can be calculated by multiplying the entries in column $C_j$ and column 'contribution per unit quantity' and adding them up. The total profit contribution in the present case is 600 x 0 + 650 x 0 + 300 x 0 = 0

3. Numbers under $X_1$ and $X_2$ are the physical ratio of substitution. For example, number 2 under $X_1$, gives the ratio of substitution between $X_1$ and $S_1$. In sample words if we wish to produce 2 units of product $P_1$ i.e., X, 2 units of $S_1$ must be sacrificed. Other numbers have similar interpretation. Similarly, the number in the identity matrix' columns $S_1$ $S_2$ and $S_3$ can be interpreted as ratios of exchange. Hence the numbers under the columns $S_1$, represents the ratio of exchange between $S_1$ and the basic variables $S_1$, $S_2$ and $S_3$.

4. $Z_j$ and $C_j – Z_j$ are the two final rows. These two rows provide us the total profit and help us in finding out whether the solution is optimal or not $Z_j$ and $C_j – Z_j$ can be found out in the following manner.

   (a) $Z_j$ = $C_j$ of $S_1$ (0) x coefficient of $X_1$ in $S_1$ row (2) + $C_j$ of $S_1$ (0) x coefficients of $X_1$ in $S_2$ row (1) + $C_j$ of $S_3$ (0) x coefficient $X_1$ in $S_3$ row (1) = 0 x 2 + 0 x 1 + 0 x 1 = 0

**Table5**

| $C_j$ | Solution mix | 8 | 12 | 0 | 0 | 0 | Contribution unit quantity |
|---|---|---|---|---|---|---|---|
| | | $X_1$ | $X_2$ | $S_1$ | $S_2$ | $S_3$ | (Solution values) |
| 0 | $S_1$ | 2 | 1 | 1 | 0 | 0 | 600 |
| 0 | $S_2$ | 1 | 2 | 0 | 1 | 0 | 650 |
| 0 | $S_3$ | 1 | 0 | 0 | 0 | 1 | 300 |
| | $C_j$ | 0 | 0 | 0 | 0 | 0 | |
| | $(C_j – C_j)$ | 8 | 12 | 0 | 0 | 0 | |

Using the same procedure $Z_j$ for all the other variable columns can be worked out as shown in the complete first Simplex Table given in Table 5.

(b) The number in the $(C_j – Z_j)$ row represent the net profit that will result from introducing 1 unit of each product or variable into the solution. This can be worked out by subtracting. $Z_j$ total for each column from the $C_j$ values at the top of that variable's column. For example, $C_j – Z_j$ number in the $X_1$ column will 8 – 0 = 8, in the $X_2$ column it will be 12 – 0 = 12 etc.

(7)     The value of the objective function can be obtained by multiplying the elements in $C_j$ column with the corresponding elements in the $C_j$ rows i.e. in the present case $Z = 8 \times 0 + 12 \times 0 = 0$

**Table 6**

| $C_j$ | Solution mix | 8 | 12 | 0 | 0 | 0 | Contribution unit quantity |
|---|---|---|---|---|---|---|---|
| | | $X_1$ | $X_2$ | $S_1$ | $S_2$ | $S_3$ | (Solution values) |
| **0** | **$S_1$** | 2 | 1 | 1 | 0 | 0 | 600 |
| 0 | $S_2$ | 1 | 2 | 0 | 1 | 0 | 650 |
| 0 | $S_3$ | 1 | 0 | 0 | 0 | 1 | 300 |
| | $C_j$ | 0 | 0 | 0 | 0 | 0 | |
| | $(C_j - C_j)$ | 8 | 12 | 0 | 0 | 0 | |

(8) By examining the number in the $(C_j - Z_j)$ row, we can see that total profit can be increased by #8 for each unit of product $X_1$ added to the product mix or by #12 for each unit of product $X_2$ added to the product mix. A positive $(C_j - Z_j)$ indicates that profits can still be improved. A negative number of $(C_j - Z_j)$ would indicate the amount by which the profits would decrease, if one unit of the variable was added to the solution. Hence, optimal solution is reached only when there are no positive numbers in $(C_j - Z_j)$ row.

**Step 5. Test for optimality**

Now we must test whether the results obtained are optimal or it is possible to carry out any improvements. It can be done in the following manner.

5.     Selecting the entering variable. We have to select which of the variables, out of the two non-basic variables $X_1$ and $X_2$, will enter the solution. We select the one with maximum value of $C_j - Z_j$ variable $X_1$ has a $(C_j - Z_j)$ value of 8 and $X_2$ has a $(C_j - Z_j)$ value of 12. Hence, we select variable $X_2$ as the variable to enter the solution mix and identify the column in which it occurs as the key column with help of a small arrow.

6.     Selecting the variable that leaves the solution. As a variable is entering the solution, we have to select a variable which will leave the solution. This can be done as follows:

(a) Divide each number in the solution value or contribution unit quantity column by corresponding number in the key column i.e. divide 600, 650 and 300 by 1, 2, 0.

**Table7**

| $C_i$ | Solution Mix | 8 $X_1$ | 12 $X_2$ | 0 $S_1$ | 0 $S_2$ | 0 $S_3$ | Solution values | Minimum ratio |
|---|---|---|---|---|---|---|---|---|
| 0 | $S_1$ | 2 | 1 | 1 | 0 | 0 | 600 | **600** |
| 0 | $S_2$ | 1 | 2 | 0 | 1 | 0 | 650 | **325** |
| 0 | $S_3$ | 1 | 0 | 0 | 0 | 1 | 300 | ∞ |
| | $Z_j$ | 0 | 0 | 0 | 0 | 0 | | |
| | $(C_j - Z_j)$ | 8 | 12 | 0 | 0 | 0 | | |

Key

↑
Key column

(b) Select the row with smallest non negative ratio as the row to be replaced, in present example the ratio are

$S_1$ row, $600/1 = 600$ unit of $X_2$

$S_2$ row, $650/2 = 325$ units of $X_2$

$S_3$ row, $300/0 = ∞$ units of $X_2$

It is clear that $S_2$ (with minimum ratio) is the departing variable. This row is called the key row.

(c)     The number at the intersection of key row and key column is called is called the key number which is 2 in the present case and is circled in the table.

### Step 6. Developing second simplex table

Now we can develop the second simplex table by the following method.

(a)     Determine new values for the key row. To revise the key rows, divide he values in the key row ($S_2$) by value of the element (2) and replace departing variable ($S_2$) by the entering variable ($X_2$).

(b)     Determine new values for other remaining rows. This is done as follows:

86

New row = old row number – [Corresponding number in key row] x [corresponding fixed ratio] where fixed ratio = old row number in key column/key number.

Now the new $S_1$ and $S_3$ row are

Row $S_1$ = 600 – 650 x 1/2 = 275

2 – 1 x 1/2 = 1.5

1 – 2 x 1/2 = 0

1 – 0 x 1/2 = 1

0 – 1 x 1/2 = 0

0 – 0 x 1/2 = 0

Row $S_2$ = 300 – 650 x 0/2 = 300

1 – 1 x 0/2 = 1

0 – 2 x 0/2 = 0

0 – 0 x 0/2 = 0

0 – 1 x 0/2 = 0

1 – 0 x 0/2 = 1

Key row $S_2$ is replaced by $X_2$ with the following elements

1/2 , 1, 0, 1/2, 0, 325

(c)     Value of $C_j$ and $C_j - Zj$ rows can be calculated as explained earlier. The new revised and improved solution table is shown below.

**Table 8 Second simplex Table**

| $C_i$ | Solution Mix | 8 $X_1$ | 12 $X_2$ | 0 $S_1$ | 0 $S_2$ | 0 $S_3$ | Solution values | Minimum ratio |
|---|---|---|---|---|---|---|---|---|
| 0 | $S_1$ | 1.5 | 0 | 1 | 0 | 0 | 275 | $\infty$ |
| 0 | $X_2$ | 1/2 | 1 | 0 | 1/2 | 0 | 325 | 325 |
| 0 | $S_3$ | 1 | 0 | 0 | 0 | 1 | 300 | $\infty$ |
| | $Z_j$ | 10 | 0 | 0 | 0 | 0 | | |
| | $(C_j - Z_j)$ | 0 | 12 | 0 | 0 | 0 | | |

Key row→ (aligned with $S_1$ row)

Key column (aligned with $X_2$ column)

$Z_j$ values are $Z = 0 \text{ x } 1.5 + 0 \text{ x } \dfrac{1}{2} \text{ x } 0 \text{ x } 1 = 0$ etc.

Minimum ratios by dividing 275, 325 and 300 by corresponding element in the key column i.e., 0, 1, 0.

$$\frac{275}{0} = \infty$$

$$\frac{325}{1} = 325$$

$$\frac{300}{0} = \infty$$

We find that the value of objective function has been improved from 0 to $\infty$. But the correct solution is not optimal as there are positive values (12) and (8) in the $(C_j - Zj)$ row. Also, since minimum ratio is 325, we select $X_2$ row to leave the solution as $X_2$ (key column) will enter the solution. The new $X_2$ (key) row will remain same as its elements 1/2, 1, 0, 1/2, 0 and 325 have to be divided by key element, i.e. (shown circled in the above able). However, row $S_1$ and $S_3$ elements will undergo change Row $S_1$ = old row number – [corresponding number in key row] x [corresponding fixed ratio]

Fixed ratio = old row number in key column/ key number = 0

It can be concluded that this problems does not have a optimal solution as $X_2$ row is to be replaced by $X_2$ row.

**Example 3** ABC Ltd produces four products $P_1$, $P_2$, $P_3$ and $P_4$. Each one of these products has to be processed on three machines X, Y, Z. The capacity of the machines and the time required to manufacture one of each type of products are shown in the table below:

| Product | Processing time for production | | |
|---|---|---|---|
| | **Machine X** | **Machine Y** | **Machine Z** |
| $P_1$ | 2 | 4 | 3 |
| $P_2$ | 3 | 2 | 2 |
| $P_3$ | 4 | 1 | 2 |
| $P_4$ | 3 | 1 | 1 |
| Capacity (hours) | 800 | 600 | 420 |

The profit contribution / unit of products $P_1$, $P_2$, $P_3$ and $P_4$ are Rs. 8, 8, 6, 4, 2 respectively.

You are required to formulate the above as a LPP and determine the optimal product mix by using simplex method.

**Solution.**

Let $x_1$, $x_2$, $x_3$ and $x_4$ be the number of units of product $P_1$, $P_2$, $P_3$ and $P_4$ respectively.

The mathematical model is as follows :

Maximise $Z = 8x_1, + 6x_2, + 4x_3 + 2x_4$.

Subject to the following constraints

$2x_1, + 3x_2, + 4x_3 + 3x_4 \leq 800$ (capacity of machine X)

$4x_1, + 2x_2, + 1x_3 + 2x_4 \leq 600$ (capacity of machine Y)

$3x_1, + x_2, + 2x_3 + x_4 \leq 420$ (capacity of machine Z)

$x_1, x_2, x_3, x_4 \geq 0$

After introducing slack variables $S_1$, $S_2$ and $S_3$ the problem can be rewritten as

Maximize $Z = 8x_1, + 6x_2, + 4x_3 + 2x_4 + 0S_1 + 0S_2 + 0S_3$

Subject to the constraints

$2x_1, + 3x_2, + 4x_3 + 3x_4. +S_1 = 800$

$4x_1, + 2x_2, + 1x_3 + 2x_4. + S_2 = 600$

$3x_1, + x_2, + 2x_3 + x_4 +S_3 \qquad = 420$

$x_1, x_2, x_3, x_4\ S_1, S_2, S_3 \qquad \geq 0$

Initial feasible solution can be obtained by putting the first simplex table can be constructed.

**Table 9**

| $C_j$ | Solution mix | 8 | 6 | 4 | 2 | 0 | 0 | 0 | Solution Value | Minimum Ratio |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $S_1$ | $S_2$ | $S_3$ | | |
| 0 | $S_1$ | 2 | 3 | 4 | 3 | 1 | 0 | 0 | 800 | 400 |
| 0 | $S_2$ | 4 | 2 | 1 | 2 | 0 | 1 | 0 | 600 | 150 |
| 0 | $S_3$ | 3 | 1 | 2 | 1 | 0 | 0 | 1 | 420 | 140 ⟶ Key row |
| | $Z_j$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | |
| | $(C_j - Z_j)$ | 8 | 6 | 4 | 2 | 0 | 0 | 0 | | |

Key column

$x_1$ is the key column

$S_3$ is the key row.

and 3 is the key number (circled in the table)

Also $x_1$ is the entering variable and $S_3$ is the outgoing variable.

We use the following row operations to get second simplex table by entering $x_1$ in to the solution and removing $S_3$ variable. $R_3$ (old)

$$R_3 \text{ (new)} = \frac{1}{3}$$

$$R_1 \text{ (new)} = R_1 \text{ (old)} - 3R_3(\text{new})$$

$$R_2 \text{ (new)} = R_2 \text{ (old)} - 2R_3(\text{new})$$

**Table 10 Second simplex table**

| $C_j$ | Solution mix | 8 | 6 | 4 | 2 | 0 | 0 | 0 | Solution Value | Minimum Ratio |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $S_1$ | $S_2$ | $S_3$ | | |
| 0 | $S_1$ | -1 | 3 | 2 | 2 | 1 | 0 | -1 | 380 | 126.7 |
| 0 | $S_2$ | 2 | $\frac{4}{3}$ | $-\frac{1}{3}$ | $\frac{4}{3}$ | 0 | 1 | $-\frac{2}{3}$ | 320 | 240 |
| 8 | $x_1$ | 1 | $\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{1}{3}$ | 0 | 0 | $\frac{1}{3}$ | 140 | 46.7 |
| $Z_j$ | 1120 | 8 | $\frac{8}{3}$ | $\frac{16}{3}$ | $\frac{8}{3}$ | 0 | 0 | $\frac{8}{3}$ | | |

Key row →

91

| $(C_j - Z_j)$ | 0 | $\dfrac{10}{3}$ | $-\dfrac{4}{3}$ | $-\dfrac{2}{3}$ | 0 | 0 | $-\dfrac{8}{3}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | |

$$\uparrow$$

Key column

$R_3$ (new) $= \dfrac{1}{3} \times 3 = 1,\ \dfrac{1}{3} \times 1 = \dfrac{1}{3},\ \dfrac{1}{3} \times 2 = \dfrac{2}{3}\ ,\ \dfrac{1}{3}, 0, 0,\ \dfrac{1}{3}, 140$

*i.e.,* $1,\ \dfrac{1}{3},\ \dfrac{2}{3},\ \dfrac{1}{3}, 0, 0,\ \dfrac{1}{3},\ 140$

$R_1$ (new) $= 2 - 3 \times 1 = -1,\ 3 - 3 \times \dfrac{1}{3} = 2,\ 4 - 3 \times \dfrac{2}{3} = 2,\ 3 - 3\ \times\ \dfrac{1}{3} = 2$

$1 - 3 \times 0 = 1,\ 0 - 3 \times 0 = 0,\ 0 - 3 \times \dfrac{1}{3} = -1,\ 800 - 3 \times 140 = 380$

*i.e.,* 1, 2, 2, 2, 1, 0, - 1, 380.

$R_2$ (new) $= 4 - 2 \times\ 1 = 2,\ 2 - 2 \times \dfrac{1}{3} = \dfrac{4}{3},\ 1 - 2 \times \dfrac{2}{3} = -\dfrac{1}{3},\ 2 - 2 \times \dfrac{1}{3} = \dfrac{4}{3}$

$0 - 2 \times 0 = 0,\ 1 - \times 0 = 1,\ 0 - 2 \times \dfrac{1}{3} = -\dfrac{2}{3},\ 600 - 2 \times 140 = 320$

*i.e.,* $2,\ \dfrac{4}{3},\ -\dfrac{1}{3},\ \dfrac{4}{3}, 0, 1,\ -\dfrac{2}{3}, 320$

Calculation of $Z_j$

$Z_j (x_1) = -1 \times 0 + 2 \times 0 + 1 \times 8 = 8$

$Z_j (x_2) = 2 \times 0 + \dfrac{4}{3} \times 0 + \dfrac{1}{3} \times 8 = \dfrac{8}{3}$

$Z_j (x_3) = \dfrac{16}{3}$

$Z_j (x_4) = \dfrac{8}{3}$ $\qquad Z_j = 380 \times 0 + 320 \times 0 + 140 \times 8 = 1120$

$$Z_j (S_1) = 0$$

$$Z_j (S_2) = 0$$

$$Z_j (S_3) = \frac{8}{3}$$

It can be seen that Z has improved from 0 to 1120 but since there is still a positive vale in $(C_j - Z_j)$ it is not optimal solution.

It is now clear that $x_2$ is the entering variable and $x_1$ the departing variable.

Now the third simplex table is o be constructed

We now use the following row operations to get a new solution by entering $x_2$ and removing $x_1$ variable.

$$R_1 \text{ (new)} = R_1 \text{ (old)}$$

$$R_2 \text{ (new)} = R_2 \text{ (old)} - \frac{2}{3} R_1 \text{ (new)}$$

$$R_3 \text{ (new)} = R_3 \text{ (old)} - \frac{2}{3} R_1 \text{ (new)}$$

$$R_1 \text{ (new)} = -1, 2, 2, 2, 1, 0, -1, 380$$

$$R_2 \text{ (new)} = 2 - \frac{2}{3} \text{ x } -1 = \frac{8}{3}, \frac{4}{3} - \frac{2}{3} \text{ x } 2 = 0, - \frac{1}{3} - \frac{2}{3} \text{ x } 2 =$$

$$\frac{4}{3} - \frac{2}{3} \text{ x } 2 = 0, 0 - \frac{2}{3} \text{ x } 1 = \frac{-2}{3}, \text{ x } 1 = \frac{-2}{3}, 0 - \frac{2}{3} \text{ x } 0 = 0$$

$$\frac{1}{3} - \frac{2}{3} \text{ x } -1 = 1, 320 - \frac{2}{3} \text{ x } 380 = \frac{200}{3}$$

i.e. $\quad \frac{8}{3}, 0, \frac{-5}{3}, 0, \frac{-2}{3}, 0, 1, \frac{200}{3}$

$$R_3 \text{ (new)} = 1 - \frac{2}{3} \text{ x } -1 = \frac{5}{3}, \frac{1}{3} - \frac{2}{3} \text{ x } 2 = -1, \frac{2}{3} - \frac{2}{3} \text{ x } 2 = \frac{-2}{3}$$

$$\frac{1}{3} - \frac{2}{3} \times 2 = -1, \quad 0 - \frac{2}{3} \times 1 = \frac{-2}{3}, \quad 0 - \frac{2}{3} \times 0 = 0$$

$$\frac{1}{3} - \frac{2}{3} \times -1 = 1, \quad 140 - \frac{2}{3} \times 380 = \frac{-340}{3}$$

i.e. $\quad \frac{5}{3}, -1, \frac{-2}{3}, -1, \frac{-2}{3}, 0, 1, \frac{-340}{3}$

**Table 11 Third simplex table**

| $C_j$ | Solution mix | 8 | 6 | 4 | 2 | 0 | 0 | 0 | Solution mix |
|---|---|---|---|---|---|---|---|---|---|
| | | $x_1$ | $X_2$ | $x_3$ | $x_4$ | $S_1$ | $S_2$ | $S_3$ | |
| 0 | $S_1$ | -1 | 2 | 2 | 2 | 1 | 0 | -1 | 380 |
| 0 | $S_2$ | $\frac{8}{3}$ | 0 | $\frac{-5}{3}$ | 0 | $\frac{-2}{3}$ | 0 | 1 | $\frac{200}{3}$ |
| 6 | $x_2$ | $\frac{5}{3}$ | -1 | $\frac{-2}{3}$ | -1 | $\frac{-2}{3}$ | 0 | 1 | $\frac{-340}{3}$ |
| | $Z_j$ | 10 | -6 | $\frac{-12}{3}$ | -6 | -4 | 0 | 6 | |
| | $(C_j - Z_j)$ | -2 | 0 | 8 | 8 | 4 | 0 | -6 | |

The student should further attempt this problem to get the optimal solution. The present solution is not the optimal solution as positive values exist in $C_j - Z_j$

### 3.2 Minimisation problems (All Constraints of the type ≥) Big 'M' Method

We have till now seen in this chapter, the type of problems where profit had to be maximized and the constraints were of the type $\leq$. However, there could be problems where the objective function has to be minimized (like the availability of funds, raw material or the costs of operations have to be minimized) and the constraints involved may be of the type $>$ or $=$. In such cases, the Simplex method is somewhat different and is discussed in the form of following steps.

**Step I.** Formulation of mathematical model

Minimise $Z = C_1 x_1 + C_2 x_2 + C_3 x_3 + \ldots\ldots\ldots\ldots + C_n x_n$

Subject to the constraints

$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \ldots\ldots\ldots a_{1n} x_n \geq b_1$

$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \ldots\ldots\ldots a_{2n} x_3 \geq b_2$

$\begin{matrix} . & & . & . & . & & . \\ . & & . & . & . & & . \\ . & & . & . & . & & . \end{matrix}$

$a_{m1} x_1 + a_{m2} x_2 + a_{m3} x_3 + \ldots\ldots\ldots a_{mn} x_n \geq b_m$

where $x_1, x_2, x_3 \ldots\ldots\ldots\ldots_n \geq 0$

Now we subtract the surplus variables $S_1$, $S_2$, $\ldots\ldots$ $S_n$ etc to convert the inequalities into equations.

*i.e.,* Minimise $Z = C_1 x_1 + C_2 x_2 + C_3 x_3 + \ldots\ldots\ldots C_n X_n + 0S_1 + 0S_2 + \ldots\ldots + 0S_n$

Subject to the constraints

$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \ldots\ldots\ldots a_{1n} x_n - S_1 = b_1$

$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \ldots\ldots\ldots a_{2n} x_3 - S_2 = b_2$

$\begin{matrix} . & & . & . & . & & . \\ . & & . & . & . & & . \\ . & & . & . & . & & . \end{matrix}$

$a_{m1} x_1 + a_{m2} x_2 + a_{m3} x_3 + \ldots\ldots\ldots a_{mn} x_n - S_m = b_m$

*where $x_i > 0$ (i = 1, 2, ...... n)*

$S_j > 0$ (j = 1, 2, ......m)

As in the maximization problem, initial basic solution is obtained by putting $x_1 = x_2 \ldots = x_n = 0$

So     $- S_1 = b_1$     or     $S_1 = - b_1$

$- S_2 = b_2$     or     $S_2 = - b_2$

. . . .
. . . .
. . . .

$- S_m = b_m$     or     $S_2 = - b_2$

It may be seen that $S_1$, $S_2$ ……….. $S_m$ being negative violate the non-negativity constraint and hence is not feasible. Hence, in the system of constraints we introduce $m$ new variables $A_1$ $A_2$…….$A_m$ known as artificial variable. By introducing these variables the equations are

$$a_{11}\, x_1 + a_{12}\, x_2 + a_{13}\, x_3 + ………… a_{1n}\, x_n - S_1 + A_1 = b_1$$

$$a_{21}\, x_1 + a_{22}\, x_2 + a_{23}\, x_3 + ………… a_{2n}\, x_3 - S_2 + A_2 = b_2$$

. . . . .
. . . . .
. . . . .

$$a_{m1}\, x_1 + a_{m2}\, x_2 + a_{m3}\, x_3 + …………. a_{m\,n}\, x_n - S_m + A_m = b_m$$

*where $x_j > 0$ (i = 1, 2, 3, …… n)*

$$S_j > 0 \; (j = 1, 2, 3, …….m)$$

$$A_j > 0 \; (j = 1, 2, 3, …….m)$$

As we have introduced artificial variables $A_1$ $A_2$……$A_m$ this has to be taken out of the solution. For this purpose, we introduce a very large value (M) assigned to each of artificial variable and zero to each of he surplus variables as the coefficient values in the objectives function. The problem now become

Minimise $Z = C_1\, x_1 + C_2\, x_2 + C_3\, x_3 + …………. C_n\, X_n$

$$+ \; 0S_1 + 0S_2 + ………+ 0S_m +$$

$$MA_1 + MA_2 + …………… + MA_m$$

Subject to constraints

$$a_{11}\,x_1 + a_{12}\,x_2 + a_{13}\,x_3 + \ldots\ldots\ldots\ a_{1n}\,x_n - S_1 + A_1 = b_1$$

$$a_{21}\,x_1 + a_{22}\,x_2 + a_{23}\,x_3 + \ldots\ldots\ldots\ a_{2n}\,x_3 - S_2 + A_2 = b_2$$

$$\begin{matrix} . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{matrix}$$

$$a_{m1}\,x_1 + a_{m2}\,x_2 + a_{m3}\,x_3 + \ldots\ldots\ldots\ a_{m\,n}\,x_n - S_m + A_m = b_m$$

Step 2. Setting up of initial simplex table

**Here, we allot 0 values to variables $x_1 = x_2 = x_3 \ldots\ldots = x_n = 0$ so that $A_1 = b_1$, $A_2 = b_2 \ldots\ldots A_m = b_m$.**

**Table 12 Initial simplex table**

| CB | Solution mix | $C_j$ Solution values | $C_1$ $C_2$ $C_3$ ..... $C_n$ 0 0 M ... M $x_1$ $x_2$ $x_3$ ..... $x_n$ $S_1$ $S_2$ $S_m$ $A_1$ $A_2$...$A_m$ | Minimum ratio |
|---|---|---|---|---|
| $CB_1$ | $A_1$ | $b_1$ | $a_{11}$ $a_{12}$ ...... $a_{1\,n}$ $-1$ $0$ $0$ $1$ $0$ ...... $0$ | |
| $CB_2$ | $A_2$ | $b_2$ | $a_{21}$ $a_{22}$ ...... $a_{2\,n}$ $0$ $-1$ $0$ $0$ $1$ ...... $0$ | |
| . | . | . | . . . . . . . . . | |
| . | . | . | . . . . . . . . . | |
| . | . | . | . . . . . . . . . | |
| $CB_n$ | $A_m$ | $B_m$ | $A_{m1}$ $a_{m2}$ ...... $a_{m\,n}$ $0$ $0$ $-1$ $0$ $0$ ...... $1$ | |
| | $Z_j$ ($C_j - Z_j$) | | $0$    $0$      $0$ $0$ $0$ $0$ $0$ ...... $0$<br><br>$C_1$   $C_2$ ..........$C_n$ $0$ $0$ $0$ M M  .... M | |

**Step 3. Test for optimality**

Calculate the elements of $(C_j - Z_j)$ row

    (a) If all $(C_j - Z_j) > 0$ then the basic feasible solution is optimal.
    (b) If anyone $(C_j - Z_j) < 0$ then pick up the largest negative number in this row. This is the key column and determines the variable entering the solution.
    Now the second simplex table can be constructed.

**Step 4. Test for feasibility**

Determine the key row and key number (element) in the same manner as is done in the maximization problem.

**Example 4** A special diet for a patient in the hospital must have at least 8000 units of vitamins, 100 units of minerals and 2800 units of calories. Two types of foods X and Y are available in the market at the cost of #8 and #6 respectively. One unit of X con tains 40 units of vitamins, 2 units of minerals and 80 units of calories. One unit of food B contains 200 units of vitamins, 4 units of minerals and 80 units of calories. What combination of foods X and Y be used so that the minimum requirement of vitamins, minerals and calories is maintained and the cost incurred by the hospital is minimized? Use simplex method.

**Solution.** Mathematical model of the problem is as follows

        Minimize $Z = 8x_1 + 6x_2$

        Subject to the constraints

        $400x_1 + 200x_2 \geq 800$ (Constraint of minimum vitamins)

            $2x_1 + 4x_2 \geq 100$ (Constraint of minimum minerals)

        $80x_1 + 80x_2 \geq 2800$ (Constraint of minimum calories)

  $x_1, x_2 \geq 0$ (Non – negativity constraint)

where $x_1$ and $x_2$ are the number of units of food X and food Y. Now the constraint inequalities can be converted into equations. Here, we take an initial solution with very high cost, as opposed to the maximum problem where we had started with an initial solution with no profit. We subtract surplus variables $S_1$, $S_2$ and $S_3$.

$400x_1 + 200x_2 - S_1 = 800$

   $2x_1 + 4x_2 - S_2 = 100$

$80x_1 + 80x_2 - S_3 = 2800$

The surplus variables $S_1$, $S_2$ and $S_3$ introduced in these equations represent the extra unit of vitamins, minerals and calories over 8000 units, 100 units and 2800 units in the least cost combinations.

Let $x_1$, $x_2$ be zero in the initial solution

Hence  $S_1 = - 8000$

$S_2 = - 100$

$S_3 = - 2800$

This is not feasible as $S_1$, $S_3$ and $S_3 \geq 0$ and  cannot be negative. We have to see hat $S_1$, $S_2$ and $S_3$ do not appear (as they are – ve) in the initial solution. So is $x_1$, $x_2$ and $S_1$, $S_2$, $S_3$ are all zero, new foods which can substitute food X and Y must be introduced. $A_1$, $A_2$ and $A_3$ are he artificial variable to be introduced. Let the artificial variables (foods) be of are large price, M per unit

$400x_1 + 200x_2 - S_1 + A_1 = 800$

$2x_1 + 4x_2 - S_2 + A_2 = 100$

$80x_1 + 80x_2 - S_3 + A_3 = 2800$

And Z object function

Minimise $Z = 8x_1 + 6x_2 + 0S_1 + 0S_2 + 0S_3 + MA_1 + MA_2 + MA_3$

where $x_1$

$x_2$, $S_1$, $S_2$, $S_3$, $A_1$, $A_2$, $A_3 \geq 0$

Now, it is possible to set up initial  solution by putting  $x_1 = x_2 = S_1 = S_2$, $= S_3$, $= 0$ in such a manner that $A_1 = 8000$,m $A_2 = 100$ and $A_3 = 2800$.


Table  13

First simplex table

|   |   | $C_j$ | 8 | 6 | 0 | 0 | 0 | M | M | M |   |
|---|---|-------|---|---|---|---|---|---|---|---|---|
|   | B | $b(=x_B)$ |   |   |   |   |   |   |   |   |   |

| $C_B$ | Solution mix variable | Solution values | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | $A_1$ | $A_2$ | $A_3$ | Min ratio |
|---|---|---|---|---|---|---|---|---|---|---|---|
| M | $A_1$ | 8000 | 400 | 200 | -1 | 0 | 0 | 1 | 0 | 0 | 20 |
| M | $A_2$ | 100 | 2 | 4 | 0 | -1 | 0 | 0 | 1 | 0 | 50 |
| M | $A_3$ | 2800 | 80 | 80 | 0 | 0 | -1 | 0 | 0 | 1 | 35 |
| | $Z_j$ | | 482M | 284 M | -M | -M | -M | M | M | M | |
| | $(C_j - Z_j)$ | | 8 – 482M | 6 - 284M | M | M | M | 0 | 0 | 0 | |

Key column

$x_1$ is the key column entering the solution, A is the departing row and 400 (circled) in the table is the key number (element).

Now apply the row operations

(i)  $R - 1$ (new) $\rightarrow \dfrac{1}{400} R - 1$ (old)

(ii) $R - 2$ (new) $\rightarrow R - 2$ (old) $- 2R - 1$ (new)

(iii) $R - 3$ (new) $\rightarrow R - 3$ (old) $- 80 R - 1$ (new)

Table 14 second simplex table

| $C_B$ | Solution mix variable (= B) | Solution values $b(= x_B)$ | $C_j$ 8 | 6 | 0 | 0 | 0 | M | M | M | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | $A_1$ | $A_2$ | $A_3$ | Min ratio |
| 8 | $x_1$ | 20 | 1 | $\dfrac{1}{2}$ | $-\dfrac{1}{400}$ | 0 | 0 | | 0 | 0 | 40 |

| M | A$_2$ | 60 | 0 | 3 | $\dfrac{1}{200}$ | -1 | 0 | | 1 | 0 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| M | A$_2$ | 1200 | 0 | 40 | $\dfrac{1}{5}$ | 0 | -1 | | 0 | 1 | 30 |
| Z$_j$ | | | 8 | 4+43 M | -4+41 M/200 | -M | -M | | M | M | |
| (C$_j$ – Z$_j$) | | | 0 | 2 - 43M | 4 – 41 M/200 | M | M | | 0 | 0 | |

Key column

Value of Z calculated as follows

$Z_j\ (x_1) = 8 \times 1 + M \times 0 = 8$

$Z_j\ (x_2) = \dfrac{1}{2} \times 8 + 3 \times M + 40M = 4 + 43M$

$Z_j\ (S_1) = \dfrac{1}{400} \times 8 + \dfrac{1}{200} M + \dfrac{1}{5} M = \dfrac{-4+41M}{200}$

$Z_j\ (S_2) = -M$

$Z_j\ (S_3) = -M$

$Z_j\ (A_2) = M$

$Z_j\ (A_3) = M$

It is clear from the above table, that $x_2$ enters the solution and A$_2$ departs, using the following row operations, we introduce $x_2$ and remove A$_2$.

(i) R – Z (new) $\rightarrow \dfrac{1}{3}$ R – 2 (old)

(ii) R – 1 (new) → R – 1 (old) – $\dfrac{1}{2}$ R – 2 (new)

(iii) R – 3 (new) → R – 3 (old) – 40 R – 1 (new)

R – 2 (new) = 20, 0, 1, $\dfrac{1}{600}$, -$\dfrac{1}{3}$, 0, $\dfrac{1}{3}$, 0.

R – 1 (new) = 10, 1, 0, -$\dfrac{1}{300}$, $\dfrac{1}{6}$, 0.

R – 3 (new) = 400, 0, 0, $\dfrac{2}{15}$, $\dfrac{40}{3}$, -1.

Table 15 second simplex table

| $C_B$ | Solution mix variable (= B) | Solution values b(= $x_B$) | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $S_3$ | $A_1$ | $A_2$ | $A_3$ | Min ratio |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $C_j$ | 8 | 6 | 0 | 0 | 0 | M | M | M | |
| 8 | $X_1$ | 10 | 1 | 0 | $-\dfrac{1}{300}$ | $\dfrac{1}{6}$ | 0 | - | - | 0 | 60 |
| 6 | $X_2$ | 20 | 0 | 1 | $\dfrac{1}{600}$ | $-\dfrac{1}{3}$ | 0 | - | - | 0 | -60 |
| M | $A_3$ | 400 | 0 | 0 | $\dfrac{2}{5}$ | $\dfrac{40}{3}$ | -1 | - | - | 1 | 30 |
| $Z_j$ | | | 8 | 6 | $\dfrac{-1+8M}{60}$ | $\dfrac{-2+4M}{3}$ | -M | - | - | M | |
| $(C_j – Z_j)$ | | | 0 | 2 - 43M | $\dfrac{1-8M}{60}$ | $\dfrac{2-4M}{3}$ | M | - | - | 0 | |

It can be seen $S_2$ has to be introduced and $A_3$ has to depart. This procedure can be adopted for further improving the solution by constructing fourth simplex Table and so on.

### 3.3 MINIMISING CASE – CONSTRAINTS OF MIXED TYPES ($\leq$ AND $\geq$)

We have seen the examples earlier where the constraints were either $>$ type or $<$ type. Both these are problems where the constraint equation could contain both types of constraints. This type of problem is illustrated with the help of an example.

**Example 4**

A metal alloy used in manufacture of rifles uses two ingredients A and B. A total of 120 units of alloy is used for production. Not more than 60 units of A can be used and at least 40 units of ingredient B must be used in the alloy. Ingredient A costs Rs. 4 per unit and ingredient B costs Rs. 6 per unit. The company manufacturing rifles is keen to minimize its costs. Determine how much of A and B should be used.

**Solution:** Mathematical formulation of the problem is

$$\text{Minimise cost } Z = 4x_1 + 6x_2$$

Subject to constraints.

$$x_1 + x_2 = 120 \quad \text{(Total units of alloy)}$$

$$x_1 \leq 60 \quad \text{(ingredient A constraint)}$$

$$x_2 \leq 40 \quad \text{(ingredient B constraint)}$$

$$x_1 , x_2 \geq 0 \quad \text{(Non-negativity constraint)}$$

where $x_1$ and $x_2$ number of units of ingredient A and B respectively. Let $x_1$ and $x_2 = 0$ *and let us introduce an artificial variable which represents a new ingredient with very high cost M.*

$$x_1 + x_2 + A_1 = 120$$

*Also* $\qquad\qquad x_1 + S_1 = 60$

*Third constraint* $\qquad x_2 - S_2 + A_2 = 40$

Now the standard form of the problem is

$$\text{Minimize } Z = 4x_1 + 6x_2 + MA_1 + 0S_1 + 0S_2 + MA_2$$

Subject to the constraints

$$x_1 + x_2 + A_1 = 120$$

$$x_1 + S_1 = 60$$

$$x_2 - S_2 + A_2 = 40$$

$$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$$

Initial basic solution is obtained by putting $x_1 = x_2 = 0$ and $S_1 = S_2 = 0$ so that $A_1$ 100, $S_1 = 60$, $A_2 = 40$

**Table 16 First simplex table**

| | | $C_j$ | 4 | 6 | M | 0 | 0 | M | Minimum ratio |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | Solution mix | Solution values | $x_1$ | $x_2$ | $A_1$ | $S_1$ | $S_2$ | $A_2$ | |
| M | $A_1$ | 120 | 1 | 1 | 1 | 0 | 0 | 0 | 120 |
| 0 | $S_1$ | 60 | 1 | 0 | 0 | 1 | 0 | 0 | - |
| M | $A_2$ | 40 | 0 | ① | 0 | 0 | -1 | 0 | 40 *Key row* |
| | $Z_j$ | | M | 2M | M | 0 | -M | M | |
| | $(C_j - Z_j)$ | | 4 – M | 6 – 2M ↑ | 0 | 0 | M | 0 | |

Key column

$6 - 2M$ is the largest negative number hence, $x_2$ will enter the solution and since 40 is the minimum ratio $A_1$ will depart.

R – 3 (New) → R – 3 (old) as key element is 1

R – 1 (New) → R – 1 (old) – R – 3 (New)

## Table 17 Second simplex table

| | $C_j$ | 4 | 6 | M | 0 | 0 | M | Minimum ratio |
|---|---|---|---|---|---|---|---|---|
| $C_B$ | Solution mix | Solution values | $x_1$ | $x_2$ | $A_1$ | $S_1$ | $S_2$ | $A_2$ | |
| M | $A_1$ | 80 | 1 | 0 | 1 | 0 | 1 | | 80 |
| 0 | $S_1$ | 60 | ① | 0 | 0 | 1 | 0 | | $60 \rightarrow$ |
| 6 | $X_2$ | 40 | 0 | 1 | 0 | 0 | -1 | | - |
| | $Z_j$ | | M | 6 | M | 0 | M – 6 | | |
| | $(C_j – Z_j)$ | | 4 – M | 0 | 0 | 0 | -M +6 | | |

Key column

R – 1 (new) = 1 – 0 = 1; 1 – 1 = 0, 1 – 0 = 1, 0 – 0 = 0, 0 – ( - 1) = 1

*i.e.,* 0, 1, 1, 0, 1, 100 -40 = 60

$x_1$ will be introduced and $Si_1$ will depart

Use the following row operations

(i)      R – 2 (new) → $R_2$ (old)

(ii)     R – 1 (new) → $R_1$ (old) – $R_2$ (new)

R – 2 (new)  = 1, 0, 0, 1, 0

R – 1 (new) = 1 – 1 = 0, 0 – 0 = 0, 1 – 0 = 1, 0 – 1 = -1, 1 – 0 =1

## Table 18 Third simplex table

| C$_B$ | Solution mix | Solution values | C$_j$ | 4 | 6 | M | 0 | 0 | M | Minimum ratio |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | x$_1$ | x$_2$ | A$_1$ | S$_1$ | S$_2$ | A$_2$ | |
| M | A$_1$ | 40 | | 0 | 0 | 1 | -1 | ① | | 40 |
| 4 | x$_1$ | 60 | | 1 | 0 | 0 | 1 | 0 | | - |
| 6 | X$_2$ | 40 | | 0 | 1 | 0 | 0 | -1 | | -40 |
| | Z$_j$ | | | 4 | 6 | M | -M+4 | M – 6 | | |
| | (C$_j$ – Z$_j$) | | | 0 | 0 | 0 | M – 4 | -M +6↑ | | |

Key column

We now introduce S$_2$ and take out A$_1$ using following row operations

R – 1 (new) → R – 1 (old)

R – 3 (new) → R – 3 (old)  + R – 1 (new)

**Table 19 Forth simplex table**

| C$_B$ | Solution mix | Solution values | C$_j$ | 4 | 6 | M | 0 | 0 | M |
|---|---|---|---|---|---|---|---|---|---|
| | | | | x$_1$ | x$_2$ | A$_1$ | S$_1$ | S$_2$ | A$_2$ |
| 0 | S$_2$ | 40 | | 0 | 0 | | -1 | 1 | |
| 4 | x$_1$ | 60 | | 1 | 0 | | 1 | 0 | |
| 6 | x$_2$ | 80 | | 0 | 1 | | -1 | 0 | |
| | Z$_j$ | | | 4 | 6 | - | -2 | 0 | |

| | | | | | | |
|---|---|---|---|---|---|---|
| $(C_j - Z_j)$ | 0 | 0 | - | 2 | 0 | |

Since all the numbers in $(C_j - Z_j)$ are either zero or positive, this is the optimal solution.

$x_1 = 60$, $x_2 = 80$ and $Z = 40 \times 60 + 6 \times 80 = \#720$

## 3.4 Maximisation case-constraints of mixed type

A problem involving mixed type of constraints in which $=$, $\geq$ and $\leq$ are involved and the objective function is to be maximized.

**Example 6.** Maximise $Z = 2x_1 + 4x_2 - 3x_3$

Subject to the constraints.

$$x_1 + x_2 + x_3 \geq 8$$

$$x_1 - x_2 \geq 1$$

$$3x_1 + 4x_2 + x_3 \geq 40$$

**Solution.** The problem can be formulated in the standard form

Mazimise $Z = 2x_1 + 4x_2 - 3x_3 + 0S_1 - MA_1 - MA_2$

Subject to constraints

$$x_1 + x_2 + x_3 + A_1 = 8$$

$$x_1 - x_2 - S_1 + A_2 = 1$$

$$3x_1 + 4x_2 + x_3 + S_3 = 40$$

$$x_1 \geq 0, \ x_2 \geq 0, \ \ S_1 \geq 0, \ S_2 \geq 0, \ A_1 \geq 0, \ A_2 \geq 0.$$

Where $A_1$ and $A_2$ are the artificial constraints, $S_1$ is the surplus variable, $S_2$ is the slack variable and M is a very large quantity. For initial basic solution: $A_1 = 8$, $A_2 = 1$, $S_2 = 40$

**Table 20 First simplex table**

| $C_B$ | Solution mix variable (B) | Solution values b $(=x_B)$ | $x_1$ | $x_2$ | $x_3$ | $S_1$ | $S_2$ | $A_1$ | $A_2$ | Minimum ratio |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $C_j$ | 2 | 4 | -3 | 0 | 0 | -M | -M | |
| - M | $A_1$ | 8 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 8 |
| - M | $A_2$ | 1 | 1 | -1 | 0 | -1 | 0 | 0 | 1 | 1 |
| 0 | $C_2$ | 40 | 3 | 4 | 1 | 0 | 1 | 0 | 0 | $\dfrac{40}{3}$ |
| | $Z_j$ | | -2M | 0 | -M | M | 0 | -M | -M | |
| | $(C_j - Z_j)$ | | 2+2M ↑ | 4 | -3+M | -M | 0 | 0 | 0 | |

Key column

This is a problem of maximization, hence we select $2 + 2M$, the largest positive number in

$(C_j - Z_j)$ $x_1$ will enter and $A_2$ will depart. Use the following row operations.

$\qquad$ R – 2 (new) $\rightarrow$ R – 2 (old)

$\qquad$ R – 1 (new) $\rightarrow$ R – 1 (old) – $R_2$ (new)

$\qquad$ R – 3 (new) $\rightarrow$ R – 3 (old) – 3 $R_2$ (new)

**Table 21 Second simplex table**

| $C_B$ | Solution mix variable (B) | Solution values b $(=x_B)$ | $x_1$ | $x_2$ | $x_3$ | $S_1$ | $S_2$ | $A_1$ | $A_2$ | Minimum ratio |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $C_j$ | 2 | 4 | -3 | 0 | 0 | -M | -M | |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| - M | $A_1$ | 7 | 0 | 2 | 1 | 1 | 0 | | -1 | $\dfrac{7}{2} \rightarrow$ |
| 2 | $x_2$ | 1 | 1 | -1 | 0 | -1 | 0 | | 1 | $-1$ |
| 0 | $S_2$ | 37 | 0 | 7 | 0 | 3 | 1 | | -3 | $\dfrac{37}{7}$ |
| $Z_j$ | | | 2 | -2M-2 | -M | -M-2 | 0 | | M+2 | |
| $(C_j - Z_j)$ | | | 0 | 6+2M | -3+M | M+2 | 0 | | -2 | |

Key column

$$R - 2 \text{ (new)} = R - 2 \text{ (old)}$$

$$R - 1 \text{ (new)} = R - 1 \text{ (old)} - R - 2 \text{ (new)}$$

$$R - 3 \text{ (new)} = 40 - 3 \times 1 = 37, \ 3 - 3 \times 1 = 0, \ 4 - 3 \times - 1 = 7$$

$$0 - 3 \times 0 = 0, \ 0 - 3 \times - 1 = 3, \ 1 - 3 \times 0 = 1, \ 0 - 3$$

x 1 = - 3

Now $x_2$ will enter as new variable and $A_1$ will depart as shown. Third simplex table can be prepared by using the following row operations.

$$R - 1 \text{ (new )} = \frac{1}{2} R - 1 \text{ (old)}$$

$$R - 2 \text{ (new)} = R - 2 \text{ (old)} + R - 1 \text{ (new)}$$

$$R - 3 \text{ (new)} = R - 3 \text{ (old)} - 7 R - 1 \text{ (new)}$$

$$R - 1 \text{ (new)} = \frac{7}{2}, 0, 1, \ \frac{1}{2}, \frac{1}{2}, 0$$

$$R - 2 \text{ (new)} = \frac{9}{2}, 1, 0, \ \frac{1}{2}, \ \frac{-1}{2}, 0$$

$$R - 3 \text{ (new)} = 37 - 7 \times \frac{7}{2} = \frac{25}{2}, \ 0 - 7 \times 0 = 0, \ 7 - 7 \times 1 = 0$$

$$0 - 7 \text{ x} \frac{1}{2} = \frac{-7}{2}, \ 3 - 7 \text{ x } \frac{1}{2} = \frac{-1}{2}, \ 1 - 7 \text{ x } 0 = 1$$

$$\frac{25}{2}, \ 0, \ 0, \ \frac{-7}{2}, \frac{-1}{2}, \ 1$$

**Table 22 Third simplex table**

| $C_B$ | Solution mix variable (B) | Solution values b $(=x_B)$ | $X_1$ | $X_2$ | $x_3$ | $S_1$ | $S_2$ | $A_1$ | $A_2$ |
|---|---|---|---|---|---|---|---|---|---|
| | | $C_j$ | **2** | **4** | **-3** | **0** | **0** | **-M** | **-M** |
| 4 | $x_1$ | $\frac{7}{2}$ | 0 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | | |
| 2 | $x_2$ | $\frac{9}{2}$ | 1 | 0 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | | |
| 0 | $S_2$ | $\frac{25}{2}$ | 0 | 0 | $-\frac{7}{2}$ | $-\frac{1}{2}$ | 1 | | |
| | $Z_j$ | | 2 | 4 | 3 | 1 | 0 | | |
| | $(C_j - Z_j)$ | | 0 | 0 | -6 | -1 | 0 | | |

Since all the entries in $C_j - Z_j$ are either 0 or negative, optimal solution has been obtained with

$$x_1 = \frac{9}{2}, \ x_2 = \frac{7}{2}, \ x_3 = 0, \ S_2 = \frac{11}{2} \text{ and } Z = 2x_1 + 4x_2 - 3x_3 + 0S_1 + 0S_2$$

$$= 9 + 14 - 0 + 0 + 0 = \#23.$$

**4.0. CONCLUSION**

Simplex methods can be used to solve a linear programming model whose number of decision variable is two or more. In standard linear programming models, for maximization problems we use the " less than or equal to sign", while for minimization problem we use the "greater than or equal to sign"

## 5.0. SUMMARY

1. Simplex methods can be used to solve a linear programming model whose number of decision variable is two or more.
2. In standard linear programming models, for maximization problems. We use the " less than or equal to sign", while for minimization problem we use the "greater than or equal to sign"
3. In the simplex method, the process of determining the main variable that can be included or the non basic variable that can be excluded is known as "change of Basis"

## 6.0. TUTOR MARKED ASSIGNMENT

Following data is available for a manufacturing company engaged in production of three item X, Y and Z

| Production | Time required in hours | | Total |
|:---:|:---:|:---:|:---:|
| | Marching | Finishing | Contribution (Rs) |
| X | 12 | 3 | 1000 |
| Y | 6 | 8 | 800 |
| Z | 8 | 6 | 400 |
| Company' s capacity | 3000 | 1500 | |

You are required to present the above data in the form of LLP to maximize the profit from the production and solve the problem using simplex method.

## 7.0. REFERENCES/FURTHER READINGS

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        Education  Association.

3.    Arowolo O.(2011); Operations Research:CIT425

4.    Statistics and Operations Research. By Debashis Dutta

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## UNIT 4:  INTEGER PROGRAMMING

Introduction

8.0.    Objective

9.0.    Main Content

3.1. LIMITATIONS OF INTEGER PROGRAMMING

3.2. METHODS OF INTEGER PROGRAMMING

3.3. INTEGER PROGRAMMING FORMULATION

## 4.0 INTRODUCTION

In mathematical programming problems, sometimes the values of the result come out to be negative and fraction, in such cases the solution is not optimal. In Linear Programming it is assumed that the decision variables can take continuous values i.e. these could be fractions or in integer. Integer Programming deals with solutions in which some or all the variables can assume integers non-negative values only. In LPP, the result may recommend the use of 4.5 machines oir employing 6.5 men, which has no meaning as fractional machines and men cannot be used. Hence,, there is a need to have a programming system where the results are always only integers and not fractions. This need is met by the Integer Programming techniques. We could have.

(a)  Pure Integer Linear Programming. If all the variables take only integer values.
(b)  Mixed Integer Linear Programming – if some of the variables are restricted to have only integer values while others could have fractional values as the case may be in real life applications of the problem.

## 5.0 OBJECTIVE

- ✓ Understand limitations of integer programming

- ✓ Identify the methods of integer programming

- ✓ Formulate integer programming model

- ✓ Solve IP problems using branch and bond method

## 6.0 MAIN TEXT

## 3.1   LIMITATIONS OF INTEGER PROGRAMMING

We have seen that in LP problems with large and complex data can be solved in a reasonable time, however the performance of integers algorithms has not been found to be uniformly efficient and useful. In integer programming rounding off is used to a value approximately true or correct. Rounding off is done in such a manner that the closet possible or nearest number is taken. If the result is 22.3 men, obviously the approximation is 22 men and not 23 men. This intersects rounding off error. This type of error or approximation may be acceptable when we talk of discrete number of variables like men, machines etc, however when we are using this algorithm for solution of financial investments, it is not rational or logical to make use of integers only. Here, in fact, it is required that exact values of money are worked  out for the best possible results.

## 3.2   METHODS OF INTEGER PROGRAMMING

The following two integer programming methods are available.

**Cutting Plane Method.** In this method of Integer Linear Programming certain 'secondary' conditions are added in such a manner that the ultimate result satisfies the conditions of only integer solutions. These 'secondary' conditions 'cut' or eliminate certain aspects of the solution which are not feasible integers. Thus the name 'cutting methods'.

**Search Methods.** Here all the possible feasible integers only are considered as the solution. The best known search method is called the branch-and-Bound techniques. A special case of each methods is when all the integer variables are binary in nature.

**Cutting-Plane Algorithm**

This method was developed by RE Gomory for pure-integer problems as also for mixed integer problems. Fractional algorithm and mixed algorithm are applied to the two problems respectively. The following steps are involved in finding the solutions.

**Step I.** Minimization problem is converted into maximization problem.

**Step II.** Solve this maximization problem without considering the  condition  of integer values

**Step III.** If the optimal solution found in step II for the variables, does not have integer values, then moves to step IV as given below.

**Step IV.** Carry out the test of integrality of the solution.

Determine the highest fraction value in solution value column of the solution. Select the row with the largest value. If there is a negative fraction, convert this into the sum of negative and a non-negative fraction. Than the row which contained the larget fraction is written in the form of an equation. Now we obtained equations with fractional parts of all coefficients by ignoring integral parts and replacing the whole number by zeros.

**Step V.** Here the technical coefficient = fractional part of a resource avialbility + some integer. Hence it is equal to or grater than the fractional part of resource availability. So, fractional part is taken to the R.H.S. and the inequation is formed as greater than or equal to $(< =)$ type. If this is to be converted into $< =$ type, it is multiplied with $-1$ and to make it as an inequality a slack is introduced.

**Step VI.** The constraint is added to the optimum simplex table of the solution found in step II. Now solve the problem by Dual simplex Method.

**Step VII.** If the solution has all integer values, then this is the optimal solution. However, if there are some fractional values, go back to step III. The procedure is repeated till an optimum solution with all the integer values is obtained. The above method will be explained with the help of examples.

### 3.3 INTEGER PROGRAMMING FORMULATION

Use the same mathematical notations as were used in he formulation of LPP, the integer programming can be mathematically written as

Maximise or optimize $Z = \sum_{j=1}^{n} C_j X_j$

Subject to the constraints

$\sum_{j=1}^{n} a_{ij} x_j \leq b_{ii} = 1, 2, 3, ...... m$

$x1 \geq 0 \ j = 1, 2, 3 .... n$

and $x_i$ integer value j = 1, 2, ......... s

The most common use of integer programming is found in the real world problems related to investment decisions, budgeting, Protection Planning and Control (PPC) in manufacturing industry, traveling salesmen etc. Some of these cases are discussed in succeeding examples.

**Example 19.6**

| Project | Investment during year | | | PV of expected return |
|---|---|---|---|---|
| | 1 | 2 | 3 | |
| P – 1 | 1000,000 | 600,00 | 500,00 | 800,00 |
| P – 2 | 500,000 | 200,00 | 400,00 | 700,00 |
| P – 3 | 300,000 | 250,00 | 350,00 | 400,00 |
| P – 4 | 400,000 | 300,00 | 260,00 | 300,00 |
| Funds for investment | 18,00,000 | 10,00,000 | 800,000 | |

*An investment consultant has four projects with different investments and present value of expected returns. Funds available for investment during the three proposal are also available. The detailed information regarding the project is as follows.*

*Formulate an integer programming model for the consultant to make a decision as to which project should be accepted in order to maximize present value of expected return.*

**Solution.**

Let $X_1$ , $X_2$, $X_3$ and $X_4$ be the investment on projects P – 1, P – 2, P – 3 and P – 4 respectively.

Maximize $Z = 8,00,000 + 7,00,00,000 \, x \, 2 + 4,00,000 \, x \, 3 + 3,00,000 \, x \, 4$

Subject to the constraints

$$10,00,000 \text{ x } 1 + 5,00,00,000 \text{ x } 2 + 3,00,000 \text{ x } 3 + 4,00,000 \text{ x } 4 \leq 18,00,000$$

$$6,00,000 \text{ x } 1 + 2,00,00,000 \text{ x } 2 + 2,50,000 \text{ x } 3 + 3,00,000 \text{ x } 4 \leq 10,00,000$$

$$5,00,000 \text{ x } 1 + 4,00,00,000 \text{ x } 2 + 3,50,000 \text{ x } 3 + 2,60,000 \text{ x } 4 \leq 8,00,000$$

Where $X_1$, $X_2$, $X_3$, $X_4 \geq 0$ and are integers.

**Example 19.7.**

*A multinational Company (MNC) is planning to invest in four different projects in Business Process Outsourcing (BPO) industry in an important town of North. The details of the investment of MNC (in thousands of naira) are provided below:*

| Project | Present value of expected returns | Capital requirement for three years | | |
|---|---|---|---|---|
| | | 1 | 2 | 3 |
| A | 800 | 600 | 500 | 550 |
| B | 550 | 900 | 400 | - |
| C | 400 | 300 | 200 | 400 |
| D | 250 | 400 | 150 | 100 |
| Funds available for investment | 1500 | 1200 | 700 | 500 |

*It is also know that projects A and B are mutually exclusive. However, Project D can only be accepted if project C is acceptable due to technology constraints. Which project should the MNC accept to maximize their present value of expected returns?*

**Solution.**

Let $X_1$, $X_2$, $X_3$, and $X_4$ be the investment in projects A, B C and D respectively. Also, let $X_i = 1$ (if project *j* is accepted) and $X_j = 0$ (if project *j* is rejected)

Maximise (PV of returns) $Z = 800 \, x \, 1 + 500 \, x \, 2 + 400 \, x \, 3 + 250 \, x \, 4 \leq 1500$

Subjective to the constraints.

$600 \, x \, 1 + 900 \, x \, 2 + 300 \, x \, 3 + 400 \, x \, 4 \leq 1200$

$500 \, x \, 1 + 400 \, x \, 2 + 200 \, x \, 3 + 150 \, x \, 4 \leq 700$

$550 \, x \, 1 + 400 \, x \, 3 + 100 \, x \, 4 \leq 500$

$X_1 + X_2 \geq 1$

$- X_3 + X_4 \leq 1$

$X_j = 0 \text{ or } 1$

## 3.4    BRANCH AND BOUND METHOD

In certain type of problems of the variables of an Integer Programming Problem (PP) have the constraint of a upper limit or a lower limit or both upper and lower bounds. The method used to solve such problem is called Branch & Bound Method and is applicable to pure as well mixed IPP.

The basic method involves dividing the feasible region into smaller sub-sets, each sub-set is considered sequentially until a feasible solution giving he optimal value of objective function is arrived at. The procedure is as given un the following steps.

**Step I.** Optimal solution of the Linear Programming problem is obtained without considering the restrictions of integer

**Step II.**  Test the integrality of the optimal solution obtained above

(a)     If the solution turns out to be in integers, then this is the optimum solution of the given IPP

(b)     If the solution is not in integers, then proceed to step II

**Step III.** Consider the upper bound values of the objective function, determine the lower bound values by rounding of to the integer values of the decision  variables.

Sub-Problem I – Given LPP with an additional constraint $x_j < [x^*_j]$

Sub-Problem II – Given LPP with an additional constraint $x_j < [x_j^*] + 1$

Where $x_j^*$ is the optium value of $x_j$ (not an integer) and $[x_j^*]$ is the largest integer contained in $x_j^*$.

**Step V.** Solve the above two sub problems. The following cases may arise.

    (a)    Optimum solution of the two sub-problem is in integers, then the solution obtained is the optimal solution.

    (b)    (b) One-sub problem – Integral
            Second – sub problem – No feasible solution

In this case, the optimum solution is that of the integral solution of sub-problem one. Second sub-problem solution is ignored.

    (c)    One sub-problem – integral
            Second sub – problem – Non – integral

In this case, repeat the steps III and IV for the second sub problem

**Step VI.** Repeat steps III to V until we get all solutions with integral values.

**Step VII.** Out of the integral value solution achieved, select the one which gives the optimum value of Z.

**Example 2**

*Min* $Z = -4x_1 + x_2 + 2x_3$

*Subject to*

$$2x_1 - 3x_2 + 2x_3 \leq 12$$

$$-5x_1 + 2x_2 + 3x_3 \geq 4$$

$$3x_1 - 2x_3 = -1$$

$$x_1, x_2, x_3 \geq 0$$

**Solution.**

In simple form, the problem reduces to

$$\text{Min } Z = -4x_1 + x_2 + 2x_3$$

Subject to $\quad 2x_1 - 3x_2 + 2x_3 + x_4 = 12$

$$-5x_1 + 2x_2 + 3x_3 - x_5 = 4$$

$$-3x_1 + 2x_3 \qquad\qquad = 1$$

$$x_1, \ x_2, \ x_3, \ x_4, \ x_5 \geq 0$$

Since the above equations do not contain basic variables, artificial variables $x_6$ and $x_7$ are added to the problem. Then the problem is

$$\text{Min } z = -4x_1 + x_2 + 2x_3 + 0x_4 + Mx_6 + Mx_7$$

Subject to

$$2x_1 - 3x_2 + 2x_3 + x_4 = 12$$

$$-5x_1 + 2x_2 + 3x_3 - x_5 = 4$$

$$-3x_1 + 2x_3 + x_7 = 1$$

Let $S_1$ to $S_7$ denote the column vectors corresponding to $x_1$ to $x_7$.

$$P_1 = \begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix} \qquad P_2 = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} \qquad P_3 = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

$$P_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad P_5 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \qquad P_6 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$P_7 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad b = \begin{bmatrix} 12 \\ 4 \\ 1 \end{bmatrix}$$

As $x_4$, $x_6$ and $x_7$ from the initial basis, we have

$$B = [P_4 \quad P_6 \quad P_7] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$$

$$B^{-1} I = b^- = B^{-1} b = b$$

The initial table of the revised simplex is given below.

| Basic variables | B⁻¹ | Solution values | Entering Variable | Pivot Column |
|---|---|---|---|---|
| $x_4$ | $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | 12 | | 2 |
| $x_6$ | | 4 | $x_3$ | 3 |
| $x_7$ | | 1 | | 2 |

The Simplex multipliers are

$$= (0 \quad M \quad M) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (0, \ M, \ M)$$

$$\therefore C_1 - XP_1 = -4 - (0, \quad M, \quad M) \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix} = 8M - 4$$

$$C_2 - XP_2 = 1 - (0, \quad M, \quad M) \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} = 1 - 2M$$

$$C_3 - XP_3 = 2 - (0, \quad M, \quad M) \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = 2 - 5M$$

$$C_5 - XP_5 = 01 - (0, \quad M, \quad M) \begin{bmatrix} 0 \\ -1 \\ \\ 0 \end{bmatrix} = M$$

As $C_3 - XP_3$ is the most negative values $X_3$ will be the entering variable.

The first solution is

$$P_3 = B^{-1} P_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

2 is the first or key element.

Applying the minimum ratio rule $\dfrac{12}{2} = 6, \quad \dfrac{4}{3} = 12, \quad \dfrac{1}{2} = 5$

Minimum ratio is of $X_7$ so it will be the outgoing variable.

## 4.0. CONCLUSION

Integral programming has limitation and this leads to integral programming rounding off.

## 5.0. SUMMARY

The limitation of integral programming was introduced. Two methods of integral programming were discussed; questions on integral programming were formulated and solved in order to drive home what has been taught in this unit.

## 6.0. TUTOR MARKED ASSIGNMENT

1. *Min* $Z = -4x_1 + x_2 + 2x_3$

*Subject to*

$2x_1 - 6x_2 + 2x_3 \leq 10$

$-4x_1 + 2x_2 + 3x_3 \geq 6$

$3x_1 - 2x_3 = -2$

$x_1, x_2, x_3 \geq 0$

## 7.0. REFERENCES/FURTHER READINGS

1. Operations Research. By Prem Kumar Gupta and D.S. Hira

2. Operations Research. ( Problem Solvers) By Research and

   Education Association.

6. Statistics and Operations Research. By Debashis Dutta

7. Operations Research by O.T. Arowolo and G.O. Lawal

**MODULE THREE  MULTI STAGE DECISION PROCESSES**

# UNIT 1:     Dynamic Programming

## 1.0. INTRODUCTION

While discussing problem and solution there of in previous chapters we  assumed that values of decision variable, remain constants over the planning period. These problems could be con as static and solution were capable only for specific situation and for a particular period of time. But, as we know, there could be many situation in which decisions variables will change with time, such situations are considered dynamic in nature. Dynamic programming techniques help in finding dynamic solution for such problems.

Dynamic programming was originated by Richard E Bellman and GB Dantzing in early 1950's. It is a quantitative technique which convert one big/large problem having many decision variables into a sequence of problem each with small number of decisions variables. Thus, a big problem which is difficult to solve can  be converted into a series of small problems, which can be easily solved it attempts to optimize multi-stage decision variables and uses the word 'programming' in the mathematical sense of selection of optimal allocation of resources. Also, the word 'dynamic' is used to indicate that the decisions are taken at a number of stages like

daily, weekly etc. dynamic programming is different from linear programming in the following ways:

(a)    It does not involve any mathematical computation like of which are done in Simplex method. It uses a multistage approach by dividing the problem in number of sequential stages.

(b)    LP gives a single stage solution. However, dynamic programming
helps in finding optimal solution over a period of time, say over a

period of six months or one year, by breaking the problem into six

or twelve one month time problems and solving each of these.

**OBJECTIVES**

✓  Understand the terminologies used in dynamic programming

✓  Identify the methods of dynamic programming

✓  Formulate dynamic programming model

✓  Solve dynamic programming problems using branch and bond method

**MAIN CONTENT**

### 3.1.    IMPORTANT TERMS USED IN DYNAMIC PROGRAMMING

1.    Stage – When a large problem is developed into various sub-problems in a sequence, these are the stages of the original problem. It is in fact, each point where the decision must be made, for example, in salesman allocation, a stage may represent a group of cities, in the case of replacement problem, each year may represent a stage.

2.    State – Specific information describing the problem at different stages with the help of variables. The variables linking two stages are called the state variables. In the salesman allocation problem replacement problem the state is the of beginning with a new machine.

3.    Principle of Optimality - Bellman's principle of optimally sates "An optimal policy (a sequence of decision) has the property that whatever he initial state and decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision". According to this principle, a wrong decision (non-optimal) taken at one stage does not mean that optimum decisions for the remaining stages cannot be taken. This can be shown diagrammatically as follows.

Stage Decision          Stage Decision          Stage Decision

where       $n$ = stage number

$S_n$ = Input to stage $n$ from stage $n + 1$

$D_n$ = Decision variable at stage $n$.

4.    Forward and Backward Recursive approach – It is the type of computation Forward or Backward depending upon whether we proceed from stage 1 to $n$ i.e. $S_1 \rightarrow S_2\,S_3 \rightarrow S_n$ or from sage $S_n$ to $S_1$ i.e. $S_n \rightarrow S_{n-1} \rightarrow S_{n-2} \rightarrow S_1$ .

## 3.2    DYNAMIC PROGRAMMING APPROACH

Numerical problems and examples can only be discussed after a student has clear understanding of the fundamental concepts of Dynamic Programming. The two most important concepts are the concept of stage. As already brought out above a large problem is broken down into a number of smaller or sub problems and each sub-problem is referred to as a stage. Every stage is a part of the decision problem and a separate decision can be taken for each stage. Stage decision is the selection of one feasible solution out of a large number of alternatives available at every stage of the problem. The stage for decision will contribute to the overall decision of the entire problem. The second very important concept is hat of the 'state' which provides the specific 'current status' conditions or parameters which may be referred to as 'state variables'.

In the overall decision-making process for the entire problem, a decision made changes the 'state of the problem' with the aim of maximizing the returns. The next stage of the problem-solving process uses the values of the state variables which are the outcome of the decision of the previous stage.

## 3.3   FORMULATIONS AND SOLUTION OF DYNAMIC PROGRAMMING PROBLEMS

The following steps are involved in this:-

**Step I.** Definition of problem variables, formulation of objectives function in terms of maximization or minimization of an objective and list the problem constraints.

**Step II.** Specific definition of stages of a multi stage decision problem. This amounts to finding out different variables and their values for each state and specifying the relationship by which the state is determined at one stage with the state and decisions at the next stage.

**Step III.** Developing optimal return function through recursion relationship. Optimal return function at stage 1 is determined as this is slightly different from the general optimal return function for other stages.

**Step IV.** Constructing tabular representation clearly showing the values and computations at each stage of the solution. The solution may be developed manually or with the help of a suitable computer software depending upon the complexity of the problem.

**Step V.** Determining optimal solution. This is done when all stages of the problem have been sequentially solve.

**Example 1**

**(Salesman Employment Smoothening Problem)**

*A manufacturing company has divided its total target market into three zones. The Company's marketing department has been collecting data regarding the deployment of salesmen and the sales made in each zones. They have realized that the sales are directly dependent upon the number of salesmen in each zone. The data collected by the company is given in the table below. For various reasons, the company has decided to retain only 9 salesmen during the next year. The problem is to determine allocation of these salesmen to three different zones so that the totals sales cab neb maximized.*

| No of Salesmen | Profits in thousands of Naira | | |
|---|---|---|---|
| | Zone 1 | Zone 2 | Zone 3 |
| 0 | 35 | 40 | 45 |
| 1 | 40 | 50 | 50 |
| 2 | 45 | 65 | 60 |
| 3 | 60 | 75 | 70 |
| 4 | 70 | 85 | 80 |
| 5 | 80 | 95 | 90 |
| 6 | 90 | 100 | 100 |
| 7 | 105 | 105 | 110 |
| 8 | 100 | 100 | 120 |
| 9 | 90 | 105 | 100 |

**Solution.**

In this problem, the solution can be obtained by step process. The problem is to allocate 9 salesmen into three marketing zones to maximize total sales and hence profits. In this problem.

Three stages are the three zones and state variables are he number of salesmen varying from 0 to 9. For zone 1, the return corresponding to deployment of different number of salesmen are as follow:

Zone 1

No of salesmen                     0     1          2    3    4    5    6
          7      8     9

Sales (in thousands of Rupees)   35        40    45   60   70    80    90      105
100    90

Let us consider zone 1 and zone 2 together Nine salesmen can be divided into two zone 1 and 2 in 10 different ways. This is shown below:

| No of Zone salesmen I | $x_1$ | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $x_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| Zone II | $f_1(x_1)$ | 90 | 100 | 105 | 90 | 80 | 70 | 60 | 45 | 40 | 35 |
| Sales (in thousands of rupees) Zone I | $f_2(x_2)$ | 40 | 50 | 65 | 75 | 85 | 95 | 100 | 10 | 100 | 105 |
| Zone II | | | | | | | | | | | |
| | Total | 130 | 150 | 170 | 165 | 165 | 165 | 160 | 150 | 140 | 140 |

Where $x_1$, $x_2$ are the salesmen n zone 1 and Zone 2 respectively

and $f_1(x_1)$ = sales from zone I

$f_2 (x_2)$ = sales from zone II

Let S = Total sales from each combination

Then S $= f_1 (9) + f_2 (0)$

$$= f_1 (8) + f_2(1)$$

$$= f_1 (7) + f_2(2)$$

$$:$$

$$:$$

$$= f_1 (0) + f_2(9)$$

In general S = $f_1 (x) + f_2(9\text{-}x)$

Or         S = $f_1 (x) + f_2(A \text{ - } x)$

Where A is the number of salesmen to be allocated to Zone 1 and Zone 2

Maximise S = F (A) $[f_1 (x) + f_2(A \text{ - } x)]$

F (A) is the maximum sales

This equation can be used to determine the optimum distribution of any number of salesmen. In the present case of 9 salesmen, the distribution in zone 1 and zone 2 is shown below.

**Fig 1**

Expected sales for all combination are provided in the table. For a particular number of allocation of salesmen, the sales can be read along the diagonal. For example if 3 salesmen are to be distributed in the two zones the sales in combination possible are 3+ 0, 2 + 1, 1 + 2, 0 + 3 and can be read along diagonal 3 – 3. Maximum profit of # 1, 10,000 results from combination of 0 salesmen for zone 1 and 3 for zone 2. Te optimum results for all combinations can be tabulated as follows.

| | Maximum Sales from Optimum allocation of Salesmen in Zone 1 and Zone 2 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| No of Salesmen A | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| Total sales $f_1(x_1) + f_2(x_2)(x_2 + x_1)$ | 75 | 85 | 140 | 110 | 120 | 130 | 135 | 145 | 155 | 170 |
| | 0 + 0 | 0 + 1 | 0 + 2 | 0 + 3 | 0 + 3 | 0 + 5 | 1 + 5 | 3 + 4 | | |

Now, we can move to the next stage and 9 -salesmen can be allotted to three zones Zone 1, Zone 2 and Zone 3. It means allotting certain salesmen to zone 3 and the balance would be allotted to Zone 1 and Zone 2 put together and then further they will be distributed between Zone 1 and Zone 2. For example, we allot 4 salesmen to Zone 1 and 2 and between 5 to Zone 3, then the best sales would be

$$S = F(4) + f_3(5)$$

where F(4) – Maximum sales by Zone 1 and Zone 2

F(5) – Maximum sales in Zone 3 if salesmen are allotted in general, it can be written as

$$S = F(x) + f_3(A - x)$$

where $x$ = Salesmen allotted to one 1 and Zone 2 combined

$(A - x)$ = Salesmen allotted to Zone 3.

*i.e.* Maximise $S = F(A) = F(x) + f_3$ 9A –x)] , $0 \leq x \leq A$

Let us use the subscript II for the first two zones *i.e.* $Z_1$ and $Z_2$

then $F_3(A_3)$ = Maximum $[F_{II}(A_{II}) + f_3(A_3 - A_{II})]$, $0 \leq x \leq A_3$

The calculations for selecting the optimum combination of $A_{II}$ and $(A_3 - A_{II})$ with $A_3 = 9$ can be carried out the same way as done earlier. It is represented in the following table

**No of Salesmen in Zone 1 + Zone 2**

**Fig 2**

It can be seen optimum combination is along diagonal 9 – 9 i.e. 7 salesmen in Zone 3 and 2 combined in Zone 1 and Zone 2. This gives maximum sales of # 250,000. Further distribution of 2 salesmen in Zone 1 and Zone 2 can be seen from our earlier table *i.e.* maximum sale of 140,000 for 2 salesmen i--Zone 2 and 0 in Zone 1.

**4.0. CONCLUSION**

This unit focuses on stage, state and principle of optimality (forward and backward recursive approach. The Dynamic programming approach, formulation and solution of Dynamic programming problems.

**5.0. SUMMARY**

This unit focuses on stage, state and principle of optimality (forward and backward recursive approach. The Dynamic programming approach, formulation and solution of Dynamic programming problems.

**6.0. TUTOR MARKED ASSIGNMENT**

1. Explain the important terms in Dynamic programming that you have leant.
2. State and explain the steps that are involved in the formulation and solution of Dynamic programming.

**7.0. REFERENCES/FURTHER READINGS**

1.      Operations Research. By Prem Kumar Gupta and D.S. Hira

2.      Operations Research. ( Problem Solvers) By Research and

    Education  Association.

14.     Statistics and Operations Research. By Debashis Dutta

15.     Operations Research By ). O.T. Arowolo and G.O. Lawal

**MODULE FOUR  SCHEDULING MODELS**

UNIT 1:    Transportation model

UNIT 2:    Assignment Model

**UNIT 1        TRANSPORTATION MODEL**

1.0.    Introduction
2.0.    Objective
3.0.    Main Content
        3.1. Transportation Problems Defined

        3.2. The General Transportation Problem

        3.3. Balanced Transportation Problem

        3.4. Unbalanced Transportation Problem

        3.5. Method of Solution

        3.6. Degeneracy and the Transportation Problem

        3.7. Testing the Solution for Optimality

        3.8. Solution of Unbalanced Transportation Problem

        3.9. Maximization and the Transportation Techniques

3.0.    Conclusion
4.0.    Summary
5.0.    Tutor Marked Assignment
6.0.    Reference/ Further Readings


**1.0. INTRODUCTION**

The transportation problem is a particular form of the general linear programming problem which is usually solved by a different technique to the simplex method. This chapter describes transportation models and shows a step by step method of solution.

The transportation models are used when a firm is trying to decide where to locate a new facility. Before opening a new warehouse, factory or sales office, it is a good practice to consider a number of alternatives sites. Good financial decision concerning facility location also attempt to minimize total transportation and product costs for the entire system.

## 2.0. OBJECTIVE
- ✓ Understand limitations of integer programming
- ✓ Identify the methods of integer programming
- ✓ Formulate integer programming model
- ✓ Solve IP problems using branch and bond method

## 3.0 MAIN CONTENT

### 3.1     Transportation Problems Defined

The typical transportation problem involves a number of sources of supply (e.g. factory) and a number of destinations (e.g. warehouses). A source or supply point is defined as having only outgoing flows, a destination or sink point is defined as having only incoming flows. The capacities or demands are assumed to be real value; as are the cost or profit coefficients.

The usual objective is to minimize the transportation cost of supplying quantities of a commodity from the source to the destination. The major requirement is that, there must be a constant transportation cost per unit. However, there are some

situations when a transportation objective is to maximize. This
will be discussed later.

**3.2    The General Transportation Problem**

The general form of a transportation problem for $'m'$ sources and
$'n'$ destination can be represented as:

 **i.**  A generalized network model
 **ii.** A transportation tableau
 **iii.** A linear programming model

**Transportation as a Generalized Network**
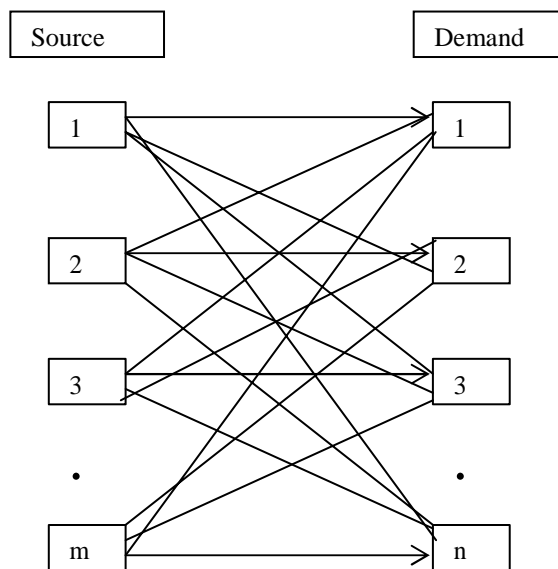


Fig. 1

Transportation Tableau

| Source | Destination | | | | | Supply |
|--------|-------------|-----|-----|-----|-----|--------|
|        | 1 | 2 | 3 | ... | $n$ | $(s_i)$ |
| 1 | $c_{11}$ | $c_{12}$ | $c_{13}$ | ... | $c_{1n}$ | $s_1$ |

136

|  | $x_{11}$ | $x_{12}$ | $x_{13}$ |  | $x_{1n}$ |  |
|---|---|---|---|---|---|---|
| 2 | $c_{21}$ $x_{21}$ | $c_{22}$ $x_{22}$ | $c_{23}$ $x_{23}$ | ... | $c_{2n}$ $x_{2n}$ | $s_2$ |
| 3 | $c_{31}$ $x_{31}$ | $c_{32}$ $x_{32}$ | $c_{33}$ $x_{33}$ | ... | $c_{3n}$ $x_{3n}$ | $s_3$ |

| . | . | . | . | ... | . | . |
|---|---|---|---|---|---|---|
| . | . | . | . |  | . | . |
| . | . | . | . |  | . | . |
| $m$ | $c_{m1}$ $x_{m1}$ | $c_{m2}$ $x_{m2}$ | $c_{m3}$ $x_{m3}$ | ... | $c_{mn}$ $x_{mn}$ | $s_m$ |
| Demand $(d_i)$ | $d_1$ | $d_2$ | $d_3$ | ... | $d_n$ | Total |

Table 1

The above general transportation tableau has the following characteristics:

i. The sources are treated as rows and destination as column.
ii. There are $(m \times n)$ cells in the tableau.
iii. The transportation cost, $c_{ij}$, from source '$i$' to destination '$j$' is recorded in the top right corner of each cell.

iv. The supply from each source is listed in the last column on the right hand side.

v. The demand from each destination is recorded at the bottom row.

vi. The $x_{ij}$ variable in each cell represent the number of units of product transported from source '$i$' to destination '$j$'.

vii. The lower right hand corner cell reflects the total supply and the total demand.

<u>Transportation as a Linear Programming Model</u>

Min $z = \sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} c_{ij}x_{ij}$ .

S.t:

$\sum\limits_{i=1}^{m} x_{ij} = d_j$ (Demand $d_j$ at destination $j$), $j = 1, 2, ..., n$.

$\sum\limits_{j=1}^{n} x_{ij} = s_i$ (Supply $s_i$ from source $i$), $i = 1, 2, ..., m$.

$x_{ij} \geq 0$ for all $i, j =$

$\sum\limits_{i=1}^{m} s_i = \sum\limits_{j=1}^{n} d_j$ (To indicate that transportation problem is balanced).

However, it should be noted that for the type of algorithm to be described here; the tabular representation will be used.

**3.3    Balanced Transportation Problem**

This is when the total supply from all sources exactly equal total demand at all destination. This type of problem is referred to as a balanced transportation problem. It is rare to observe a balanced transportation problem in reality, however the analysis of a balanced problem is a good starting point to understanding of the transportation solution processes.

For an illustration, see the table below:

**EXAMPLE 1**

Consider the transportation problem with the following unit costs and capacities.

| Source | Destination | | | Supply |
|--------|-----|-----|-----|--------|
| | *A* | *B* | *C* | |
| 1 | 5 | 1 | 6 | 200 |
| 2 | 8 | 4 | 3 | 350 |
| 3 | 7 | 9 | 5 | 170 |
| Demand | 200 | 300 | 200 | |

Table 2

From the table above, you will notice that the row total (i.e. supply) is exactly equal to the column total (i.e. demand), which is 720 units.

**3.4    Unbalanced Transportation Problem**

When the sum of the row requirement is not equal to the sum of the column requirements, the transportation problem is said to be unbalanced. There are two possibilities:

1) <u>Over Production</u>

   If the sum of the row requirement (supply) is greater than the sum of the column requirements, (demand), the production at the factories or sources exceeds the demand at the destination, warehouses or sinks and a condition of over production exists.

   The transportation problem can be balanced by creating an artificial destination, warehouse or sink where the excess units are sent. This is equivalent to adding one column to the cost and distribution matrices. This additional column is given a requirement equal to difference between the sum of rows and column requirements.

   For the purpose of this book, the cost coefficients will be assumed to be all zeros. Their true values depend upon the situation at hand. However, they could have positive values equal to the cost of storing the excess inventory at each destination.

   For example, consider a problem with the following cost matrix with row and column requirements as an example of over production.

   **EXAMPLE 2**

   Consider the transportation problem with the following unit costs and capacities.

| Plant | Warehouse | Monthly capacity |
|-------|-----------|------------------|
|       |           |                  |

|  | *A* | *B* | *C* |  |
|---|---|---|---|---|
| 1 | 20 | 19 | 21 | 12200 |
| 2 | 19 | 22 | 18 | 2800 |
| 3 | 20 | 20 | 20 | 2500 |
| 4 | 21 | 20 | 19 | 2200 |
| Monthly demand | 7300 | 3500 | 5600 |  |

Table 3

The over production is $3300$ units. With the addition of an artificial column (destination) called "Dummy", the

balanced transportation problem becomes:

| Plant | Warehouse | | | | Monthly capacity |
|---|---|---|---|---|---|
| | *A* | *B* | *C* | Dummy | |
| 1 | 20 | 19 | 21 | 0 | 12200 |
| 2 | 19 | 22 | 18 | 0 | 2800 |
| 3 | 20 | 20 | 20 | 0 | 2500 |
| 4 | 21 | 20 | 19 | 0 | 2200 |
| Monthly demand | 7300 | 3500 | 5600 | 3300 | |

Table 4

2) Under Production

When the sum of row requirements is less than the sum of column requirements, the demand at the destinations (sinks) exceeds the production at the factories (sources) and under production exists. To balance the problem, add an artificial factory (source) with a scheduled production equal to the unsatisfied demand. Again, for the purpose of this book, the cost coefficients will be assumed to be all zeros. Their true values depend upon the situation at hand.

The following is an example of under production with the cost matrix, row and column requirement.

**EXAMPLE 3**

Consider the transportation problem with the following unit costs and capacities.

| Source | Destination | | | | | Supply |
|---|---|---|---|---|---|---|
| | *A* | *B* | *C* | *D* | *E* | |
| *X* | 100 | 50 | 90 | 30 | 130 | 30000 |
| *Y* | 90 | 30 | 70 | 50 | 110 | 20000 |
| *Z* | 95 | 30 | 75 | 40 | 120 | 20000 |
| Demand | 10000 | 12000 | 15000 | 17000 | 25000 | |

Table 5

The level of under production is 9000 units. To balance the problem, add an artificial row (factory, supply or source) with a scheduled production of 9000 units. The balanced transportation is:

| Source | Destination | | | | | Supply |
|---|---|---|---|---|---|---|
| | *A* | *B* | *C* | *D* | *E* | |
| *X* | 100 | 50 | 90 | 30 | 130 | 30000 |
| *Y* | 90 | 30 | 70 | 50 | 110 | 20000 |
| *Z* | 95 | 30 | 75 | 40 | 120 | 20000 |
| Dummy | 0 | | | 0 | 0 | 9000 |
| Demand | 10000 | 12000 | 15000 | 17000 | 25000 | |

**3.5**

Table 6

**Method of Solution**

As mentioned in the introduction, linear programming can be used to solve this type of problem. However, more efficient special purpose algorithm has been developed for the transportation application. As in the simplex algorithm, they involve finding an initial feasible solution and the making step by step improvements until an optimal solution is reached. Unlike the simplex method, the transportation methods are fairly simple in terms of computation.

Here, we will take a look at the following methods of solution which gives an initial feasible solution to transportation problem.

The methods are:

  i.   North-west corner method;
 ii.   Least cost first method;
iii.   Vogel's approximation method.

While stepping stone method and the modify improved method (MDI) are iterative technique for moving from an initial feasible solution to an optimal solution.

It must be mentioned however that before any of these method can be applied, the transportation problem must be a balanced one.

## North-west Corner Method

This method advocates that allocation should be made on the basis of geographical location of the cells in the tableau. In particular, the method attaches greater importance to the cell situated at the upper left hand corner of the tableau and makes as much as possible an allocation to the cell with both the supply restriction and demand constraint taking into consideration.

The algorithm for North-west corner methods are:

i. Exhaust the supply (source) capacity at each row before moving down to the next row.
ii. Exhaust the demand (destination) requirements of each column before moving to the right of the next column.
iii. Continue in the same manner until all supply has been exhausted and demand requirements have been met.

**EXAMPLE 4**

A firm has three factories in Lagos, Ibadan and Benin which make weekly dispatches to four depots located at Kaduna, Kano, Kebbi and Katsina. The transport cost per cost of goods dispatch along route is shown in the table below as well as the weekly quantities available from each factory and the requirement of each depot.

Transport cost / create

| Storage | Demand point | | | | Supply capacity |
|---|---|---|---|---|---|
| | Kaduna | Kano | Kebbi | Katsina | |
| Lagos | 5 | 4 | 5 | 6 | 100 |
| Ibadan | 3 | 3 | 6 | 6 | 200 |
| Benin | 2 | 5 | 7 | 8 | 400 |
| Demand | 200 | 100 | 150 | 250 | |

Table 7

How should the product be allocated to the depots? Use the North-west corner method for the initial allocation.

**SOLUTION**

Using the procedure describe above for the North-west corner method. The table below shows the initial allocation.

| Storage | Demand point | | | | Supply capacity |
|---|---|---|---|---|---|
| | Kaduna | Kano | Kebbi | Katsina | |
| Lagos | 5 100(1) | 4 | 5 | 6 | 100 |
| Ibadan | 3 100(2) | 3 100(3) | 6 | 6 | 200 |
| Benin | 2 | 5 | 7 150(4) | 8 250(5) | 400 |
| Demand | 200 | 100 | 150 | 250 | |

Table 8

Note

The numbers in the table represent deliveries and the numbers in the brackets (1) , (2) , etc. represent the order of allocation (distribution). The number in the top right corner of each cell represent unit cost of transportation.

Transportation Cost for this Initial Allocation in using North-west Corner Method

| Route | | Unit shipped | × | Cost per unit | = | Total cost |
|---|---|---|---|---|---|---|
| | | | | (#) | | (#) |
| From | To | | | | | |
| Lagos | Kaduna | 100 | × | 5 | = | 500 |
| Ibadan | Kaduna | 100 | × | 3 | = | 300 |
| Ibadan | Kano | 100 | × | 3 | = | 300 |
| Benin | Kebbi | 150 | × | 7 | = | 1050 |
| Benin | Katsina | 250 | × | 8 | = | 2000 |
| | | | | | | 4150 |

The total cost from North-west corner method is #4,150 for the initial feasible solution.

Least Cost Method

The method advocates that allocation should be based on minimum cost of transportation rule / criterion. It says that the first allocation must be made to the cell with the most minimum cost of transportation per unit. In other words, we look at the scheduled of the transport cell and identify the most minimum.

After identifying the cell with the least transportation cost, we next make maximum allocation to the cell without violating both supply and demand restriction.

To demonstrate the use of least cost first method, consider the problem of Example 4.

| Storage | Demand point | | | | Supply capacity |
|---------|--------------|------|-------|--------|-----------------|
|         | Kaduna | Kano | Kebbi | Katsina | |
| Lagos | 5 | 4 | 5 100(3) | 6 | 100 |
| Ibadan | 3 | Table 9 100(2) | 6 50(4) | 6 50(5) | 50 |
| Benin | 2 200(1) | 5 | 7 | 8 200(6) | 400 |
| Demand | 200 | 0 | 50 | 250 | |

The transportation cost for this initial allocation is:

| Route | | Unit shipped | × | Cost per unit (#) | = | Total cost (#) |
|---|---|---|---|---|---|---|
| From | To | | | | | |
| Benin | Kaduna | 200 | × | 2 | = | 400 |
| Ibadan | Kano | 100 | × | 3 | = | 300 |
| Lagos | Kebbi | 100 | × | 5 | = | 500 |
| Ibadan | Kebbi | 50 | × | 6 | = | 300 |
| Ibadan | Katsina | 50 | × | 6 | = | 300 |
| Benin | Katsina | 200 | × | 8 | = | 1600 |
| | | | | | | 3400 |

The total cost from least cost first method is #3,400 for the initial feasible solution.

Note that allocation 5 can be before 4 and vice versa.

So, for alternative solution, allocation 5 will be performed before.

<u>Alternative Solution</u>

| Storage | Demand point | | | | Supply capacity |
|---|---|---|---|---|---|
| | Kaduna | Kano | Kebbi | Katsina | |
| Lagos | <u>5</u> | <u>4</u> | <u>5</u><br>100(3) | <u>6</u> | 100 |

| | | | | | |
|---|---|---|---|---|---|
| Ibadan | <u>3</u> | <u>3</u><br><br>100(2) | <u>6</u> | <u>6</u><br><br>100(4) | 50 |
| Benin | <u>2</u><br><br>200(1) | <u>5</u> | <u>7</u><br><br>50(5) | <u>8</u><br><br>150(6) | 150 |
| Demand | 200 | 0 | 500 | 250 | |

Table 10

The transportation cost for this initial allocation is:

| Route | | Unit<br>shipped | × | Cost<br><br>per unit | = | Total cost<br><br>(#) |
|---|---|---|---|---|---|---|
| From | To | | | (#) | | |
| Benin | Kaduna | 200 | × | 2 | = | 400 |
| Ibadan | Kano | 100 | × | 3 | = | 300 |
| Lagos | Kebbi | 100 | × | 5 | = | 500 |
| Ibadan | Katsina | 100 | × | 6 | = | 600 |

| Benin | Kebbi | 50 | × | 7 | = | 350 |
|-------|-------|-----|---|---|---|------|
| Benin | Katsina | 150 | × | 8 | = | 1200 |
| | | | | | | 3350 |

## Vogel's Approximation Method

In his consideration, Vogel felt that rather than basing allocation on least cost alone; the penalty or opportunity cost one we pay by taking a wrong decision would have been an additional criterion for allocation. Hence, he felt a combination of least cost and opportunity cost would be better for allocation purpose and thus, he introduce the idea of row and column penalties or opportunity cost. The row or column penalty cost is computed by identifying the two $(2)$ least cost in each row and each column and then find the difference.

Whatever result obtained is taking as penalty for that row or column. If two costs in a row or column are tied for the rank of least cost, the penalty is zero.

The algorithms for Vogel's approximation are:

i.  Determine the penalty for each row and column.
ii. After the penalties have been calculated for all rows and columns, locate the greatest; whether a row or a column

penalty and place the variable in the cell that has the least cost in the row or column with the greatest penalty. The value of the variable is set equal to the smaller of the row and column requirements corresponding to the variable being brought into the solution. The row or column whose requirement is satisfied is deleted from further consideration and the requirement of the other (row or column) is reduced by the value assigned to the variable entering the solution.

iii. If a row requirement has been satisfied, the column penalties must be recomputed because the elements of the cost matrix corresponding to the row deleted are no longer considered in the calculation of column penalties.

iv. If a column requirement has been satisfied, the row penalties must be recomputed because the element of the cost matrix corresponding to the column deleted are no longer considered in the calculation of row penalties.

v. If a tie develops between two or more row or column penalties, select the least cost cell among these two or more rows and columns. If however, a further tie occurs among the least cost cell, then, select arbitrary (using good judgment) which among the tied cost cell will be used.

vi. Repeat steps $I$ through $VI$ until the solution is completed.

Although, it cannot be proved mathematically that Vogel's method yields a near optimal solution. Frequently, the North-west corner method and the least cost first method yields an initial solution which is far from optimal.

Application of this method will be demonstrated with the problem of Example 4.

**EXAMPLE 6**

| Storage | Demand points | Supply capacity |
|---------|---------------|-----------------|
|         |               |                 |

|  | Kaduna | Kano | Kebbi | Katsina |  |
|---|---|---|---|---|---|
| Lagos | 5 | 4 | 5 | 6 | 100 |
| Ibadan | 3 | 3 | 6 | 6 | 200 |
| Benin | 2 | 5 | 7 | 8 | 400 |
| Demand | 200 | 100 | 150 | 250 |  |

Table 11

**SOLUTION**

Cycle1

Step1: Compute the row and column penalties on the cost matrix.

| Storage | Demand points | | | | Row penalties |
|---|---|---|---|---|---|
|  | (200) Kaduna | (100) Kano | (150) Kebbi | (250) Katsina |  |
| Lagos | 5 | 4 | 5 | 6 | 1 |

| | | | | | |
|---|---|---|---|---|---|
| (100) | | | | | |
| Ibadan (200) | 3 | 3 | 6 | 6 | 0 |
| Benin (400) | 2 | 5 | 7 | 8 | 3 ← |
| Column penalties | 1 | 1 | 1 | 0 | |

Table 12

The greatest penalty is $3$ (marked with arrow). This occurs in the $3rd$ row (i.e. Benin row). Therefore, the first variable will be entered in the $3rd$ row of the cell with least cost in this row.

Step 2

With reference to the distribution table of cycle1, we assign a maximum value from row 3 (Benin) to column1(Kaduna). Here, demand is 200 units and supply is 400 units, so that maximum value is 200 units with excess supply of 200 units to be shipped to another destination. Thus, we have:

| Storage | Demand points | | | |
|---|---|---|---|---|
| | (0) Kaduna | (100) Kano | (150) Kebbi | (250) Katsina |
| Lagos (100) | <u>5</u> | <u>4</u> | <u>5</u> | <u>6</u> |
| Ibadan (200) | <u>3</u> | <u>3</u> | <u>6</u> | <u>6</u> |
| Benin (200) | (1) <u>2</u> <br> 200 | <u>5</u> | <u>7</u> | <u>8</u> |

Table 13

Note that Kaduna (i.e. column1) demand has been satisfied. Therefore, it is eliminated from further consideration.

Rows1, 2 and 3 penalties must be recomputed.

 Cycle 2

Step1: Compute new row penalties.

| Storage | Demand points | | Row |
|---|---|---|---|
| | | | |

|  | (0) Kaduna | (100) Kano | (150) Kebbi | (250) Katsina | penalties |
|---|---|---|---|---|---|
| Lagos (100) | 5 | 4 | 5 | 6 | 1 |
| Ibadan (200) | 3 | 3 | 6 | 6 | 3 ← |
| Benin (200) | (1) 2   200 | 5 | 7 | 8 | 2 |
| Column penalties | - | 1 | 1 | 0 | |

Table 14

The greatest penalty is $3$ (marked with arrow). This occurs in the $2nd$ row (i.e. Ibadan). Therefore, the second variable will be entered in the $2nd$ row of the cell with least cost in this row.

Step $2$

With reference to the distribution table of cycle $2$, we have two least cost in the $2nd$ row (i.e. Ibadan). They are: column $1$ (Kaduna) and column $2$ (Kano). But, the demand at column $1$ (Kaduna) has been satisfied which now leave us with column $2$ (Kano) only. We now assign a maximum value from Row $2$ (Ibadan) to column $2$ (Kano). Here, demand is $100$ units and supply

is 200 units, so the maximum value is 100 units (demand) with excess supply of 100 units to be shipped to another destination. Thus, we have:

| Storage | Demand points | | | |
|---|---|---|---|---|
| | (0) Kaduna | (0) Kano | (150) Kebbi | (250) Katsina |
| Lagos (100) | 5 | 4 | 5 | 6 |
| Ibadan (100) | 3 | (2) 3 100 | 6 | 6 |
| Benin (200) | (1) 2 200 | 5 | 7 | 8 |

Table 15

Note that Kano (i.e. column 2) demand has been met. Hence, we eliminated it from further consideration.

Rows 1, 2 and 3 penalties must be recomputed.

Step 1: Compute new row penalties.

| Storage | Demand points | | | | Row penalties |
|---|---|---|---|---|---|
| | (0) Kaduna | (0) Kano | (150) Kebbi | (250) Katsina | |
| Lagos (100) | 5 | 4 | 5 | 6 | 1 ← |
| Ibadan (100) | 3 | (2)  3  100 | 6 | 6 | 0 |
| Benin (200) | (1)  2  200 | 5 | 7 | 8 | 1 ← |
| Column penalties | - | - | 1 ↑ | 0 | |

Table 16

A tie has occur here; the penalties in column 3, rows 1 and 3.

We now look at the cells in each of these column and rows altogether and select the cell with the least cost.

This occurs in row 1 (Lagos) intersection column 3 (Kebbi). Therefore, the third variable will be entered in the $1st$ row, $3rd$ column.

<u>Step</u> 2

With reference to the distribution table of cycle 3, the third variable is entered in the $1st$ row (Lagos), $3rd$ column (Kebbi).

The demand here is 150 units and supply is 100 units. We assign 100 units with excess demand of 50 units to be supplied from another source. Thus, we have:

| Storage | Demand points | | | |
|---|---|---|---|---|
| | (0) Kaduna | (0) Kano | (50) Kebbi | (250) Katsina |
| Lagos (0) | <u>5</u> | <u>4</u> | (3) <u>5</u> 100 | <u>6</u> |
| Ibadan (100) | <u>3</u> | (2) <u>3</u> 100 | <u>6</u> | <u>6</u> |
| Benin (200) | (1) <u>2</u> 200 | <u>5</u> | <u>7</u> | <u>8</u> |

Table 17

Note that Lagos (row 1) supply has been exhausted. We eliminated this row from further consideration.

Columns 3 and 4 penalties must be recomputed.

Step1: Compute new column penalties.

| Storage | Demand points | | | | Row penalties |
|---|---|---|---|---|---|
| | (0) Kaduna | (0) Kano | (50) Kebbi | (250) Katsina | |
| Lagos (0) | $\underline{5}$ | $\underline{4}$ | (3) $\underline{5}$ 100 | $\underline{6}$ | - |
| Ibadan (100) | $\underline{3}$ | (2) $\underline{3}$ 100 | $\underline{6}$ | $\underline{6}$ | 0 |
| Benin (200) | (1) $\underline{2}$ 200 | $\underline{5}$ | $\underline{7}$ | $\underline{8}$ | 1 |
| Column penalties | - | - | 1 | 2 ↑ | |

Table 18

The greatest penalty is $2$ (marked with arrow). This occurs in the $4th$ column (Katsina). Therefore, the fourth variable will be entered in the fourth column of the cell with least cost in this column.

Step 2

With reference to the distribution table of cycle $4$, the $4th$ variable is entered in the $4th$ column, $2nd$ row.

The demand here is $250$ units and supply is $100$ units. We assign $100$ units with excess demand of $150$ units to be supplied from another source. Thus, we have:

| Storage | Demand points | | | |
|---|---|---|---|---|
| | (0) | (0) | (50) | (150) |
| | Kaduna | Kano | Kebbi | Katsina |
| Lagos (0) | 5 | 4 | (3)  5<br>100 | 6 |
| Ibadan (0) | 3 | (2)  3<br>100 | 6 | (4)  6<br>100 |
| Benin (200) | (1)  2<br>200 | 5 | 7 | 8 |

Table 19

Note that Ibadan (row 2) supply has been exhausted.

We eliminated this row from further consideration. Supply can only be from Benin (row 3) at this stage. Demands are from Kebbi (columns 3) and Katsina (columns 4) only. Based on this, we do not need to calculate penalties anymore. We now allocate to the least cost first. With reference to table in step 2 of cycle 4, we supply from Benin (row 3) to Kebbi (columns 3) and Katsina (columns 4) 50 units and 150 units respectively. Thus, we have the final table which looks like this:

| Storage | Demand points |
|---|---|
| | |

|  | (0) Kaduna | (0) Kano | (0) Kebbi | (0) Katsina |
|---|---|---|---|---|
| Lagos (0) | 5 | 4 | (3)  5  100 | 6 |
| Ibadan (0) | 3 | (2)  3  100 | 6 | (4)  6  100 |
| Benin (0) | (1)  2  200 | 5 | 7  50 | (5)  8  150 |

Table 20

Transportation cost for this initial allocation is:

| Route | | Unit shipped | × | Cost per unit (#) | = | Total cost (#) |
|---|---|---|---|---|---|---|
| From | To | | | | | |
| Lagos | Kebbi | 100 | × | 5 | = | 500 |
| Ibadan | Kano | 100 | × | 3 | = | 300 |
| Ibadan | Katsina | 100 | × | 6 | = | 600 |
| Benin | Kaduna | 200 | × | 2 | = | 400 |
| Benin | Kebbi | 50 | × | 7 | = | 350 |
| Benin | Katsina | 150 | × | 8 | = | 1200 |
| | | | | | | 3,350 |

We should note here that, the methods (i.e. North-west corner, Least cost first and Vogel's approximation) that we have just discuss are only meant for the initial allocation.

They might not give us the optimal allocation. In most cases, they do not. This now lead us to getting the optimal solution; but, before then, we shall discuss degeneracy.

### 3.6    Degeneracy and the Transportation Problem

The total number of allocation to be made in a transportation problem should be equal to one less than the number of rows added to the number of column i.e. Total number of allocation = Number of rows + Number of columns − 1.

On the occasions, the number of allocations turns out to be less than this (i.e. rows + columns − 1). The condition is known as degeneracy.

<u>Dealing with Degeneracy</u>

If degeneracy occurs in the allocation of a transportation problem, then it is necessary to make one or more zero allocations to routes to bring up the number of allocation to ROWS + COLUMNS − 1.

### 3.7    Testing the Solution for Optimality

By optimality test, we want to check the initial feasible solution obtained representing the minimum cost possible. This is done by

calculating what are known as 'shadow costs' (i.e. an imputed cost of not using a particular route) and comparing these with the real transport costs to see whether a change of allocation is desirable. This is done as follows with reference to the initial feasible solution of the table below:

| Source | Demand | | | | Supply |
|---|---|---|---|---|---|
| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | |
| $S_1$ | 8 4 | 6 | 10 | 15 | 4 |
| $S_2$ | 12 | 9 6 | 7 6 | 8 | 12 |
| $S_3$ | 13 2 | 13 | 10 2 | 7 10 | 14 |
| Demand | 6 | 6 | 8 | 10 | |

Table 21

Step1

Check that number of allocation is ROWS + COLUMN − 1.

Else, treat as degeneracy.

For this solution; Rows = 3 and Column = 4 .

$\therefore$ Number of allocation $= 3 + 4 - 1 = 6$, which is the same as the number of allocation made.

<u>Step 2</u>

Calculate a nominal 'Sending' and 'Receiving' cost for each occupied cell by making assumption that, the transport cost per unit is capable of being split between Sending and Receiving costs i.e.

$S_1 + D_1 = 8$

$S_2 + D_2 = 9$

$S_2 + D_3 = 7$

$S_3 + D_1 = 13$

$S_3 + D_3 = 10$

$S_3 + D_4 = 7$

Where $S_1, S_2$ and $S_3$ represent sending cost from source $S_1, S_2$ and $S_3$ and $D_1, D_2, D_3$ and $D_4$ represent receiving cost at destination $D_1, D_2, D_3$ and $D_4$.

By convention, the first source is assigned the value of zero i.e. $S_1 = 0$ and this value is substituted in the first equation and then all the other values can be obtained thus:

$D_1 = 8$, $D_2 = 7$, $D_3 = 5$, $D_4 = 2$.

Using these values, the shadow costs of the unoccupied cells can be calculated. The unoccupied cells are: $S_1 D_2, S_1 D_3, S_1 D_4$,

$S_2 D_1, S_2 D_4$ and $S_3 D_2$.

Therefore;

| Cell | Shadow cost |
|------|-------------|
| $S_1 D_2$ : | $S_1 + D_2 = 0 + 7 = 7$ |
| $S_1 D_3$ : | $S_1 + D_3 = 0 + 5 = 5$ |
| $S_1 D_4$ : | $S_1 + D_4 = 0 + 2 = 2$ |
| $S_2 D_1$ : | $S_2 + D_1 = 2 + 8 = 10$ |
| $S_2 D_4$ : | $S_2 + D_4 = 2 + 2 = 4$ |
| $S_3 D_2$ : | $S_3 + D_2 = 5 + 7 = 12$ |

These computed shadow costs are compared with the actual transport costs (from table ..). Where the ACTUAL costs are less than SHADOW costs, overall costs can be reduced by allocating units into that cell.

By comparison, we mean the difference between Actual cost and Shadow cost. This difference is sometimes referred to as the improvement index i.e.

IMPROVEMENT INDEX = ACTUAL COST − SHADOW COST.

| Cell | Actual cos | Shadow cost | Improvement index |
|------|-----------|-------------|-------------------|
| $S_1 D_2$ : | 6 | 7 | $-1$ |
| $S_1 D_3$ : | 10 | 5 | 5 |
| $S_1 D_4$ : | 15 | 2 | 13 |
| $S_2 D_1$ : | 12 | 10 | 2 |

| $S_2D_4:$ | 8 | 4 | 4 |
| $S_3D_2:$ | 13 | 9 | 4 |

The meaning of this is that, if all the improvement indices computed is greater than or equal to zero, an optimal solution has been reached. If not, it is possible to improve the current solution and decrease total transportation costs. In other words, if any of the indices is negative, an improved solution is possible. If however, there are more than one negative improvement index, our strategy would be to choose the route (unused route) with the largest negative index. Situations do arise when the largest negative index is not unique. Now, let us continue with our illustration example. The total cost could be reduced by #1 for every unit that can be transferred into cell $S_1D_2$. As there is a cost reduction that can be made, the solution in Table 21 is not optimum.

Step 3

Make the maximum possible allocation of deliveries into the cell with the (largest) negative improvement index using occupied cells i.e. $S_1D_2$ from step 2 . The number that can be allocated is governed by the need to keep within the row and column totals. This is done as follows:

| Source | Demand | | | | Supply |
|--------|--------|--------|--------|--------|--------|
| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | |
| $S_1$ | (−)  $\underline{8}$   4 | (+) $\underline{6}$ | $\underline{10}$ | $\underline{15}$ | 4 |
| $S_2$ | $\underline{12}$ | (−)  $\underline{9}$   6 | (+)  $\underline{7}$   6 | $\underline{8}$ | 12 |
| $S_3$ | (+)  $\underline{13}$   2 | $\underline{13}$ | (−)  $\underline{10}$   2 | $\underline{7}$   10 | 14 |
| Demand | 6 | 6 | 8 | 10 | |

Table 22

Table 22 is a reproduction of Table 21 with a number of + and − inserted. These were inserted for the following reasons:

Cell $S_1D_2$ : + indicates a transfer IN as indicated in step 2 .

Cell $S_1D_1$ : − indicates a transfer OUT to maintain Row $S_1$ total.

Cell $S_2D_2$ : $-$ indicates a transfer OUT to maintain Row column

      $D_2$ total.

Cell $S_2D_3$ : $+$ indicates a transfer IN to maintain Row $S_2$ total.

Cell $S_3D_1$ : $+$ indicates a transfer IN to maintain column $D_1$ total.

Cell $S_3D_3$ : $-$ indicates a transfer OUT to maintain Row $S_3$ and

      column $D_3$ balance.

The maximum number that can be transferred into cell $S_1D_2$ is the lowest number in the minus cells i.e. cells $S_1D_1$ , $S_2D_2$ and $S_3D_3$ which is 2 units.

$\therefore$ 2 units is transferred in the $+$ and $-$ sequence described above resulting in the following table:

| Source | Demand | | | | Supply |
|--------|--------|--------|--------|--------|--------|
| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | |
| $S_1$ | 8<br>2 | 6<br>2 | 10 | 15 | 4 |
| $S_2$ | 12 | 9<br>4 | 7<br>8 | 8 | 12 |
| $S_3$ | 13<br>4 | 13 | 10 | 7<br>10 | 14 |

| Demand | 6 | 6 | 8 | 10 | |
|---|---|---|---|---|---|

Table 23

The total cost of this solution is:

Cell $S_1D_1$    2 units × # 8 = # 16

Cell $S_1D_2$    2 units × # 6 = # 12

Cell $S_2 D_2$    4 units × # 9 = # 36

Cell $S_2 D_3$    8 units × # 7 = # 56

Cell $S_3D_1$    4 units × # 13 = # 52

Cell $S_3D_4$    10 units × # 7 = # 70

# 242

The new total cost is # 2 less than the total cost established in step 1. This is the result expected because it was established in step 2 that #1 would be saved for every units we were able to transfer to cell $S_1 D_2$ and we were able to transfer 2 units only.

Note: Always commence the $(+)$ and $(-)$ sequence with a $(+)$ in the cell indicated by the improvement index (actual cost $-$ shadow cost) calculation. Then, put a $(-)$ in the occupied cell in the same row which has an occupied cell in its column. Proceed until a $(-)$ appears in the same column as the original $(+)$.

Step 4

Repeat step 2 i.e. check that solution represents minimum cost (optimal). Each of the process in step 2 are repeated using the latest solution (Table 23) as a basis, thus:

Nominal dispatch and reception costs for each occupied cell.

$$S_1 + D_1 = 8$$

$$S_1 + D_2 = 6$$

$$S_2 + D_2 = 9$$

$$S_2 + D_3 = 7$$

$$S_3 + D_1 = 13$$

$$S_3 + D_4 = 7$$

Setting $S_1$ at zero, the following values are obtained:

$D_1 = 8$, $D_2 = 6$, $D_3 = 4$, $D_4 = 2$, $S_1 = 0$, $S_2 = 3$, $S_3 = 5$.

Using these values, the shadow costs of the unoccupied cells are calculated. The unoccupied cells are: $S_1D_3$, $S_1D_4$, $S_2D_1$, $S_2D_4$, $S_3D_2$ and $S_3D_3$.

| Cell | Shadow cost |
|------|-------------|
| $S_1D_3$: | $S_1 + D_3 = 0 + 4 = 4$ |
| $S_1D_4$: | $S_1 + D_4 = 0 + 2 = 2$ |
| $S_2D_1$: | $S_2 + D_1 = 3 + 8 = 11$ |
| $S_2D_4$: | $S_2 + D_4 = 3 + 2 = 5$ |
| $S_3D_2$: | $S_3 + D_2 = 5 + 6 = 11$ |
| $S_3D_3$: | $S_3 + D_3 = 5 + \dfrac{4}{4} = 9$ |

The computed shadow costs are compared with actual costs to see if any reduction in cost is possible.

| Cell | Actual cost | – | Shadow cost | = | Improvement index |
|------|-------------|---|-------------|---|-------------------|
| $S_1D_3$ | 10 | – | 4 | = | $+6$ |
| $S_1D_4$ | 15 | – | 2 | = | $+13$ |
| $S_2D_1$ | 12 | – | 1 | = | $+11$ |
| $S_2D_4$ | 8 | – | 5 | = | $+3$ |
| $S_3D_2$ | 13 | – | 1 | = | $+12$ |
| $S_3D_3$ | 10 | – | 9 | = | $+1$ |

It will be seen that all the … are positive, therefore no further cost reduction is possible and optimum has been reached.

Optimum Solution

2 units $\quad S_1 \rightarrow D_1$

2 units $\quad S_1 \rightarrow D_2$

4 units $\quad S_2 \rightarrow D_2$

8 units $\quad S_2 \rightarrow D_3$

4 units $\quad S_3 \rightarrow D_1$

10 units $\quad S_3 \rightarrow D_4$

This solution is shown in the following tableau:

|        | $D_1$ | $D_2$ | $D_3$ | $D_4$ |
|--------|-------|-------|-------|-------|
| $S_1$  | 2     | 2     |       |       |
| $S_2$  |       | 4     | 8     |       |
| $S_3$  | 4     |       |       | 10    |

Table 24

<u>Note</u>

In this example, only one iteration was necessary to produce an optimum solution mainly because a good initial solution was chosen. The principles explained above would, of course, be equally suitable for many iterations.

The total cost of this solution is:

$$\begin{array}{lll} & \# & \# \\ \text{Cell } S_1D_1 & 4 \text{ units} \times 8 & = 32 \\ \text{Cell } S_2D_2 & 6 \text{ units} \times 9 & = 54 \\ \text{Cell } S_2D_3 & 6 \text{ units} \times 7 & = 42 \\ \text{Cell } S_3D_1 & 2 \text{ units} \times 13 & = 26 \\ \text{Cell } S_3D_4 & 2 \text{ units} \times 10 & = 20 \\ \text{Cell } S_4D_4 & 10 \text{ units} \times 7 & = \underline{70} \\ & & \underline{244} \end{array}$$

### 3.8 Solution of Unbalanced Transportation Problem

Unbalanced transportation problem are solved in the same way a balanced transportation problem is handled. The method of finding an initial solution and improving on the solution by calculating Shadow cost and Actual cost are the same.

The only difference is the order in which the allocations are made. Usually, irrespective of whether you have a dummy row or column, allocation are first made to real cells according to the rules of the method of solution in used. It is after the allocation of the real cells have been taking care of, that the excess supply are allocated to the dummy column cell or excess demand allocated to dummy row cell.

**EXAMPLE 5**

A company presently operates three manufacturing plants that distribute a product to four warehouses. Currently, the capacity of the plants and the demands of the warehouses are stable. These are listed with the unit shipping costs in the following table.

Find the optimal distribution plan for the company.

| Plants | Warehouses | | | | Monthly capacity |
|---|---|---|---|---|---|
| | *A* | *B* | *C* | *D* | |
| *X* | 3 | 16 | 9 | 2 | 40 |
| *Y* | 1 | 9 | 3 | 8 | 20 |
| *Z* | 4 | 5 | 2 | 5 | 50 |
| Monthly demand | 25 | 25 | 42 | 8 | 110 / 100 |

**SOLUTION**

Table 25

Step1

Add a DUMMY destination to Table 25 with a zero transport costs and a requirement equal to the surplus availability.

$\therefore$ Dummy requirement $= 110 - 100 = 10$ products

| Plants | Warehouses | | | | | |
|---|---|---|---|---|---|---|
| | *A* | *B* | *C* | *D* | Dummy | |
| *X* | | | | | | 40 |
| *Y* | | | | | | 20 |
| *Z* | | | | | | 50 |
| | 25 | 25 | 42 | 8 | 10 | |

Step 2

Now, that the quantity available equals the quantity required (because of the insertion of the dummy), we then select one of the methods for initial feasible solution.

For the purpose of this book, the three methods described in this book will be demonstrated.

Hence, an initial feasible solution for

### i. North-west Corner Method

| Plants | Warehouses | | | | | |
|--------|-----|-----|-----|-----|-------|-----|
| | $A$ | $B$ | $C$ | $D$ | Dummy | |
| $X$ | $\underline{3}$ 25(1) | $\underline{16}$ 15(2) | $\underline{9}$ | $\underline{2}$ | $\underline{0}$ | 40 |
| $Y$ | $\underline{1}$ | $\underline{9}$ 10(3) | $\underline{3}$ 10(4) | $\underline{8}$ | $\underline{0}$ | 20 |
| $Z$ | $\underline{4}$ | $\underline{5}$ | $\underline{2}$ 32(5) | $\underline{5}$ 8(6) | $\underline{0}$ 10(7) | 50 |
| | 25 | 25 | 42 | 8 | 10 | |

The numbers in the table represent the allocations made and the numbers in the brackets represent the sequence they were inserted based on the top left corner (North-west corner) necessity to maintain row/column totals. The residue of 10 was allocated to the dummy.

**The costs of this allocation are**:

#    #

$X \to A$       25 units $\times$ 3  = 75

$X \to B$       15 units $\times$ 16 = 240

$Y \to B$       10 units $\times$ 9  = 90

$Y \to C$       10 units $\times$ 3  = 30

$Z \to C$       32 units $\times$ 2  = 64

$Z \to D$       8 units  $\times$ 5  = 40

$Z \to$ Dummy   10 units $\times$ 0  = 0

$$539$$

## ii. Least Cost First Method

| Plants | Warehouses | | | | | |
|---|---|---|---|---|---|---|
| | $A$ | $B$ | $C$ | $D$ | Dummy | |
| $X$ | 5(4) | 17(6) | | 8(3) | 10(7) | 40 |
| $Y$ | 20(1) | | | | | 20 |
| $Z$ | | 8(5) | 42(2) | | | 50 |
| | 25 | 25 | 42 | 8 | 10 | |

The numbers in the table represent the allocations made and the numbers in the brackets represent the sequence they were inserted based on lowest cost and the necessity to maintain row/column totals. The residue of 10 was allocated to the dummy.

**The costs of this allocation are**:

$$
\begin{array}{lll}
 & & \# \quad\quad \# \\
X \to A & \text{5 units} & \times\ 3\ = 15 \\
X \to B & \text{17 units} & \times\ 16 = 272 \\
X \to D & \text{8 units} & \times\ 2\ = 16 \\
X \to \text{Dummy} & \text{10 units} & \times\ 0\ = 0 \\
Y \to A & \text{20 units} & \times\ 1\ = 20 \\
Z \to B & \text{8 units} & \times\ 5\ = 40 \\
Z \to C & \text{42 units} & \times\ 2\ = \underline{84} \\
 & & \quad\quad\quad \underline{447}
\end{array}
$$

### iii. **Vogel's Approximation Method**

| Plants | Warehouses | | | | | |
|---|---|---|---|---|---|---|
| | $A$ | $B$ | $C$ | $D$ | Dummy | |
| $X$ | 25(3) | | | 8(2) | 7(7) | |
| $Y$ | | | 17(5) | | 3(6) | |
| $Z$ | | 25(1) | 25(4) | | | |
| | | | | | | |

The numbers in the table represent the allocations made and the numbers in the brackets represent the sequence they were inserted based on the penalties (greatest) and lowest cost in the row/column penalty selected necessity to maintain row/column totals. The residue of 10 was allocated partially to the dummies in row $X$ and $Y$ respectively.

**The costs of this allocation are**:

|  |  | # | # |
|---|---|---|---|
| $X \rightarrow A$ | 25 units | $\times 3$ | $= 75$ |
| $X \rightarrow D$ | 8 units | $\times 2$ | $= 16$ |
| $X \rightarrow$ Dummy | 7 units | $\times 0$ | $= 0$ |
| $Y \rightarrow C$ | 17 units | $\times 3$ | $= 51$ |
| $Y \rightarrow$ Dummy | 3 units | $\times 0$ | $= 0$ |
| $Z \rightarrow B$ | 25 units | $\times 5$ | $= 125$ |
| $Z \rightarrow C$ | 25 units | $\times 2$ | $= \underline{50}$ |
|  |  |  | $\underline{317}$ |

### 3.9 Maximization and the Transportation Techniques

Although, transportation problems are usually minimizing problems, on occasions, problems are framed so that the objective is to make the allocations from sources to destinations in a manner which maximizes contribution or profit. These problems are dealt with relatively easily as follows.

Initial Allocation

a) North-west Corner Method
   The procedure is exactly the same as that of minimization problem, because allocation is based on geographical location.

b) Least Cost First Method

For a maximization problem, this method will now be known as the greatest profit/contribution first method. The initial feasible allocation is made on basis of maximum profit/contribution first, then next highest and so on.

c) Vogel's Approximation
For a maximization problem, the penalty is the difference between the two greatest profit/contribution in rows/columns. Then select the least penalty and allocate to the cell with the greatest profit/contribution of the row/column with the least penalty selected.

Optimality Test

For optimum, the difference between the actual and the shadow profit/contribution for the unused routes should be ALL NEGATIVE. If not, make allocation into cell with the GREATEST positive improvement index.

**EXAMPLE 6 (Maximization Problem)**

A ladies fashion shop wishes to purchase the following quantities of winter dresses:

| Dress size | *I* | *II* | *III* | *IV* |
|---|---|---|---|---|
| Quantity | 100 | 200 | 450 | 150 |

Three manufacturers are willing to supply dresses. The quantities given below are the maximum they are able to supply of any given combination of orders for dresses:

| Manufacturer | *A* | *B* | *C* |
|---|---|---|---|
| Total quantity | 150 | 450 | 250 |

The shop expects the profit per dress to vary with the manufacturer as given below:

| Manufacturer | Sizes | | | |
|---|---|---|---|---|
| | *I* | *II* | *III* | *IV* |
| *A* | #2.50 | #4.00 | #5.00 | #2.00 |
| *B* | #3.00 | #3.50 | #5.50 | #1.50 |
| *C* | #2.00 | #4.50 | #4.50 | #2.50 |

You are required to:

a) Use the transportation technique to solve the problem of how the orders should be placed on the manufacturers by the fashion shop in order to maximize profit and,

b) Explain how you know that there is no further improvement possible, showing your workings.

**SOLUTION**

a) The total requirements of the shop are:
$100 + 200 + 450 + 150 = 900$ dresses,

but the total availability is only

$150 + 450 + 250 = 850$ dresses.

Accordingly, a DUMMY manufacturer capable of providing $50$ dresses must be included so that the table balances. The DUMMY will be given zero profit per dress. In this example, the initial allocation is made to give maximum profit and this results in the following table.

| Manufacturer | Sizes | Quantity available |
|---|---|---|
| | | |

|  | I | II | III | IV |  |
|---|---|---|---|---|---|
| A | (3) 2.5 \ 100 | 4 | 5 | (4) 2 \ 50 | 150 |
| B | 3 | 3.5 | (1) 5.5 \ 50 | 1.5 | 450 |
| C | 2 | (2) 4.5 \ 200 | 4.5 | (3) 2.5 \ 50 | 250 |
| Dummy | 0 | 0 | 0 | (5) 0 \ 50 | 50 |
| Quantity required by shop | 100 | 200 | 450 | 150 | 900 \ 900 |

Table 26

Note

a. The top right hand figure in each square represents the actual profits as given in the question.
b. The number in the middle of each square represents the allocation of dress sizes to manufacturers and the small bracket number represents the sequence of allocation e.g.

Cell B/III shows $450(1)$ i.e. an allocation of $450$ dresses made first because it has the highest profit per dress, #5.50

c. It will be seen that six allocations have been made. It will be recalled that a DEGENERATE situation exists if the number of allocations is less than (number of column + number of rows − 1). A DEGENERATE situation exists in this example because there should be $7$ allocations i.e.

$(4 + 4 - 1)$ but only 6 have been made. Accordingly, a zero allocation must be made and so that the shadow profits can be calculated. Cell A/III will be deemed to be occupied with a zero allocation.

d. As stated, the zero allocation must be made so that the shadow profits can be calculated. Shadow profits can be calculated when there is a linkage in the shadow profit calculations. It will be seen from the Table 26 that there is no overlapping allocation with Cell B/III from the A row. Accordingly, the zero allocation is made in Cell A/III, so Cell B/III can be evaluated. If, for example, the zero allocation had been made in Cell A/II, Cell B/III could be have been evaluated because neither B nor III was linked.

The next step is to test the initial allocation for optimality by calculating the shadow profits and comparing these with actual profits to see whether any improvement to the initial allocation can be made.

**Shadow Profit Calculation**

Occupied Cells

Cell $A/I$ = #2.5 profit

Cell $A/III$ = #5 profit

Cell $A/IV$ = #2 profit

| Cell | $B / III$ | = | #5.5 profit | |
|------|-----------|---|-------------|---|
| Cell | $C / II$ | = | #4.5 profit | (i.e. the zero allocation cell) |
| Cell | $C / IV$ | = | #2.5 profit | |
| Cell | Dummy/ $IV$ | = | 0 | Setting $A = 0$, the following values |

can be calculated:

$A = 0$, $B = 0.5$, $C = 0.5$, Dummy $= -2$, $I = 2.5$, $II = 4$, $III = 5$, $IV = 2$.

These values are used to calculate the shadow profits of the unused routes i.e.

| Cell | $A / I$ | = | #4 |
|------|---------|---|-----|
| | $B / I$ | = | #3 |
| | $B / II$ | = | #4.5 |
| | $C / I$ | = | #3 |
| | $C / III$ | = | #5.5 |
| | Dummy/ $I$ | = | #0.5 |
| | Dummy/ $II$ | = | #2 |
| | Dummy/ $III$ | = | #3 |

The shadow profits are compared with the actual profits resulting in the following table.

| Manufacturer | I | II | III | IV | Quantity available |
|---|---|---|---|---|---|
| A | 2.5 | 4      4 | 5 | 2 | 150 |
|  | 100 | 0 | 0 | 50 |  |
| B | 3      3 | 4.5      3.5 | 5.5 | 2.5   −1.0 | 450 |
|  | 0 | −1.0 | 450 |  |  |
| C | 3      2 | 4.5 | 5.5      4.5 | 2.5 | 250 |
|  | −1.0 | 200 | −1.0 | 50 |  |
| Dummy | 0.5      0 | 2      0 | 3      0 | 0 | 50 |
|  | −0.5 | −2.0 | −3 | 50 |  |
| Quantity required by shop | 100 | 200 | 450 | 150 | 900 / 900 |

Table 27

Notes

i.  The top right hand figure in each square represents actual profit.
ii. The top left hand figure in each square represents the shadow profit as calculated.
iii. The bottom right hand figures in each square are the differences between actual and shadow profits.
iv. A study of the profit differences shows that the initial allocation is optimum.

**The allocations which yield maximum profit are**:

| | | | | | |
|---|---|---|---|---|---|
| Manufacturer *A* | 100 size *I* Dresses | Profit | = | #250 |
|  | 50 size *IV* Dresses | Profit | = | #100 |
| Manufacturer *B* | 450 size *III* Dresses | Profit | = | #2,475 |

| | | | | | |
|---|---|---|---|---|---|
| Manufacturer $C$ | 200 size $II$ Dresses | Profit | = | #900 | |
| | 50 size $IV$ Dresses | Profit | = | #125 | |
| | | Total Profit | = | #3,850 | |

b) No improvement to this profit is possible because of the Actual/shadow profit comparisons given in Table 27. However, where there is a zero difference between actual and shadow profit, this indicates that an alternative solution exists giving the same profits. For example, some dresses could be ordered from Manufacturer $A$ in Sizes II and III and from Manufacturer $B$ in Size I.

## 4.0. CONCLUSION

The transportation problem is a linear programming problem that determination of the minimum cost plan for transporting goods and services from multiple sources to multiple destinations. Three methods have been used to find the initial feasible solution namely (i) the north west corner method (ii) The least cost methods (iii) The Vogel's approximation method.

## 5.0. SUMMARY

1.  The North West corner method is the easiest method and the least cost method is the improvement of it while the Vogel's approximation method is the best.

2. The second phase produces optimum solution to the transportation problem.

3. It is desired that the number of cells having items allocated to them in the initial feasible solution be equal to M + N − 1 where M is the number of rows and N is the number of columns. If this criterion is not met then degeneracy occurs.

4. For the unbalance case we usually add a dummy variable to make up for the short fall in total supply or total demand. The cost in the dummy calls is given O values.

5. The objective of the transportation problems is to minimize total transportation cost.

6. There are linear programme packages that can be used in solving a transportation problem after a formulation of the mathematical model must have been done.

7. The Optimum solution can be found using the stepping stone method and the modified distribution (MODI) method.

## 6.0. TUTOR MARKED ASSIGNMENT

1. A firm of office equipment suppliers has three depots located in various towns. It receives orders for a total of 150 special filing cabinets available and the management wishes to minimize delivery costs by dispatching the filing cabinets from the appropriate depot for each customer. Details of the availabilities, requirements and transport costs per filing cabinet are given in the table below:

| Depot | Customer | | | | Total |
|---|---|---|---|---|---|
| | $A$ | $B$ | $C$ | $D$ | |
| $X$ | #13 | #11 | #15 | #20 | 20 |
| $Y$ | #17 | #14 | #12 | #13 | 60 |

| | | | | | |
|---|---|---|---|---|---|
| Z | #18 | #18 | #15 | #12 | 70 |
| Total | 30 | 30 | 40 | 50 | 150 |

Determine the total number of filing cabinets that should be supplied from each depot and its distribution to minimize total transportation costs.

2. The weekly output figures for four factories I, II, III and IV are $220$, $100$, $65$ and $40$ units respectively. Five distributors A, B, C, D and E requires $190$, $80$, $55$, $60$ and $40$ units respectively per
week. Transport cost (in # per unit) from each factory to each

distributor are given by the following table.

| | A | B | C | D | E |
|---|---|---|---|---|---|
| I | 1 | 3 | 2 | 4 | 5 |
| II | 4 | 1 | 2 | 3 | 5 |
| III | 5 | 4 | 1 | 3 | 2 |
| IV | 2 | 4 | 5 | 6 | 6 |

a. Using the 'Least Cost First Method' to give the first solution, obtain the allocation which minimizes transport cost
b. Comment on the uniqueness of the solution. What is the alternative solution?

## 7.0. REFERENCES/FURTHER READINGS

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**UNIT 2 : ASSIGNMENT MODEL**

**4.      INTRODUCTION**

An important variation of the transportation problem is the assignment   problem. The assignment problem refers to a special class of linear programming problems that involve determining the most efficient assignment of people to projects, salesmen to territories, contracts to bidders, jobs to machines and so on. The objective is most often to minimize total costs or total time of performing the task at hand. One important characteristic of assignment problem is that only one job (or work) is assigned to one machine or  project. Because of its simple structure, the assignment problem can be solved most efficiently by its own unique method rather than by the previously described transportation methods.

**5.    OBJECTIVE**

✓ Understand the concept of assignment model

✓ Solve assignment by enumeration method

✓ Solve assignment by Hungarian method

**3.0. MAIN TEXT**

**3.1    Assignment Table**

The first step in solving an assignment problem is to develop an opportunity cost table or matrix associated with it. Generally, the rows will contain the objects or people we wish to assign and the columns comprise the tasks or things we want them assigned to. The numbers in the table are costs associated with each particular assignment.

**3.2    Method of Solving Assignment Problems**

There are basically two methods of solving the assignment problems:

i. Enumeration of Solution Method
The Enumeration method solve assignment problem by trying to see different permutation (arrangement) possible. Thereafter, we select the arrangement that gives the minimum cost. In case of a maximization problem, the arrangement that gives the maximum contribution will be selected. This method is better described by example. Below is an example to illustrate the Enumeration method.

**EXAMPLE 1**

A company employs service engineers based at various locations throughout the country to service and repair their equipment installed in customers' premises. Four requests for service have been received and the company finds that four engineers are available. The distance each of the engineers is from the various customers is given in the following table and the company wishes to assign engineers to customers to minimize the total distance to be traveled.

| Service Engineers | Customers | | | |
|:---:|:---:|:---:|:---:|:---:|
| | *W* | *X* | *Y* | *Z* |
| *A* | 25 | 18 | 23 | 14 |
| *B* | 38 | 15 | 53 | 23 |
| *C* | 15 | 17 | 41 | 30 |
| *D* | 26 | 28 | 36 | 29 |

Table 1

**SOLUTION**

<u>Step 1</u>

Reduce each column by the smallest figure in that column. The smallest figures are 15 , 15 , 23 and 14 and deducting these values from each element in the columns produces the following table.

|   | W | X | Y | Z |
|---|---|---|---|---|
| A | 10 | 3 | 0 | 0 |
| B | 23 | 0 | 30 | 9 |
| C | 0 | 2 | 18 | 16 |
| D | 11 | 13 | 13 | 15 |

Table 2

## Step 2

Reduce each row by the smallest figure in that row. The smallest figures are $0$ , $0$ , $0$ and $11$ and deducting these values gives the following table.

|   | W | X | Y | Z |
|---|---|---|---|---|
| A | 10 | 3 | 0 | 0 |
| B | 23 | 0 | 30 | 9 |
| C | 0 | 2 | 18 | 16 |
| D | 0 | 2 | 2 | 4 |

Table 3

## Note

Where the smallest value in a row is zero (i.e. as in rows A, B and C above) the row is, of course, unchanged.

## Step 3

Cover all the zeros in Table 3 by the MINIMUM POSSIBLE number of lines. The lines may be horizontal or vertical.

|   | ~~W~~ | X | Y | Z |   |
|---|---|---|---|---|---|
| A | ~~10~~ | ~~3~~ | ~~0~~ | ~~0~~ | Line 2 |
| B | ~~23~~ | ~~0~~ | ~~30~~ | ~~9~~ | Line 3 |
| C | ~~0~~ | 2 | 18 | 16 |   |
| D | ~~0~~ | 2 | 2 | 4 |   |

Line 1          Table 4

## Note

Line 3 , covering row B, could equally well have been drawn covering column X.

## Step 4

Compare the number of lines with the number of assignments to be made (in this example, there are 3 lines and 4 assignments). If the number of lines EQUALS the number of assignments to be made, go to Step 6. If the number of lines is LESS than the number

of assignments to be made (i.e. as in this example which has three lines and four assignments) then:

  i. Find the smallest UNCOVERED element from Step 3, called $x$ (in Table 4, this value is $2$).

  ii. Subtract $x$ from every element in the matrix.


  iii. Add back $x$ to every element covered by a line. If an element is covered by two lines, for example, cell A: W in Table 4, $x$ is added twice.


<u>Note</u>

The effect of these steps is that $x$ is subtracted from all uncovered elements, elements covered by one line remain unchanged, and elements covered by two lines are increased by $x$.


Carrying out this procedure on Table 4 produces the following result:

In Table 4, smallest element is $2$. The new table is:

|   | W | X | Y | Z |
|---|---|---|---|---|
| A | 12 | 3 | 0 | 0 |
| B | 25 | 0 | 30 | 9 |
| C | 0 | 0 | 16 | 14 |
| D | 0 | 0 | 0 | 2 |

Table 5

It will be seen that cells A:W and B:W have been increased by $2$ ; cells A:X, A:Y, A:Z, B:X, B:Y, B:Z, C:W and D:W are unchanged, and all other cells have been reduced by $2$.

## Step 5

Repeat Steps 3 and 4 until the number of lines covering the zeros equals the number of assignments to be made. In this example, covering the zeros in Table 5 by the minimum number of lines equals the number of assignments without any further repetition, thus:

|   | *W* | *X* | *Y* | *Z* |          |
|---|-----|-----|-----|-----|----------|
|   |     |     |     |     | Line 1   |
| *A* | 12 | 3 | 0 | 0 |          |
| *B* | 25 | 0 | 30 | 9 | Line 2   |
| *C* | 0 | 0 | 16 | 14 |          |
|   |     |     |     |     | Line 4   |
| *D* | 0 | 0 | 0 | 2 |          |

Table 6

## Step 6

When the number of lines EQUALS the number of assignments to be made, the actual assignments can be made using the following rules:

i.   Assign to any zero which is unique to BOTH a column and a row.
ii.  Assign to any zero which is unique to a column OR a row.
iii. Ignoring assignments already made, repeat rule (ii) until all assignments are made.

Carrying out this procedure for our example, results in the following:

i.   (Zero unique to BOTH a column and a row). None in this example.
ii.  (Zero unique to column OR row). Assign B to X and A to Z.
The position is now as follows:

| | W | X | Y | Z |
|---|---|---|---|---|
| $A$ | row | Satisfied | | Column satisfied |
| $B$ | row | Satisfied | | Column satisfied |
| $C$ | Table 7 | Colu satisfied | 16 | Column satisfied |
| $D$ | 0 | Column satisfied | 0 | Column satisfied |

iii. Repeat rule (ii), results in assigning D to Y and C to W.

## Notes

i.   Should the final assignment not be to a zero, then more lines than necessary were used in Step 3.
ii.  If a block of $4$ or more zero's is left for the final assignment, then a choice of assignments exists with the same distance.

## Step 7

Calculate the total distance of the final assignment.

| | | |
|---|---|---|
| $A$ to $Z$ | Distance | 14 |
| $B$ to $X$ | " | 15 |
| $C$ to $W$ | " | 15 |
| $D$ to $Y$ | " | <u>36</u> |
| | | <u>80</u> |

**EXAMPLE 2**

Example 1 will be used with the changed assumption that the figures relate to contribution and not distance and that it is required to maximize contribution.

**SOLUTION**

In each case, the Step number corresponds to the solution given for Example 1).

Original data.

| | $W$ | $X$ | $Y$ | $Z$ |
|---|---|---|---|---|
| $A$ | 25 | 18 | 23 | 14 |
| $B$ | 38 | 15 | 53 | Contributions to be gained |
| $C$ | 15 | 17 | 41 | |
| $D$ | 26 | 28 | 36 | 29 |

Table 8

Step 1

Reduce each column by LARGEST figure in that column and ignore the resulting minus signs.

|   | W | X | Y | Z |
|---|---|---|---|---|
| A | 13 | 10 | 30 | 16 |
| B | 0 | 13 | 0 | 7 |
| C | 23 | 11 | 12 | 0 |
| D | 12 | 0 | 17 | 1 |

Table 9

## Step 2

Reduce each row by SMALLEST figure in that row.

|   | W | X | Y | Z |
|---|---|---|---|---|
| A | 3 | 0 | 20 | 6 |
| B | 0 | 13 | 0 | 7 |
| C | 23 | 11 | 12 | 0 |
| D | 12 | 0 | 17 | 1 |

Table 10

## Step 3

Cover zeros by minimum possible number of lines.

|   | W | X | Y | Z |
|---|---|---|---|---|
| 3 | 0 | 20 | 6 | ? B ?0? |
|   |   |   |   | B |
| C | 23 | 11 | 12 | 0 |

| D | 12 | ~~0~~ | 17 | ~~1~~ |
|---|----|----|----|----|

Table 11

## Step 4

If the number of lines equals the number of assignments to be made, go to Step 6. If less (as in example), carry out the 'uncovered element' procedure previously described. This results in the following table:

|   | W | X | Y | Z |
|---|---|---|---|---|
| A | 0 | 0 | 17 | 6 |
| B | 0 | 16 | 0 | 10 |
| C | 20 | 11 | 9 | 0 |
| I | Table 12 | 0 | 14 | 1 |

## Step 5

Repeat Steps 3 and 4 until number of lines equals the number of assignments to be made. In this example, this occurs without further repetition, thus:

|   | W | X | Y | Z |
|---|---|---|---|---|

| | | | | |
|---|---|---|---|---|
| A | 0̶ | 0̶ | 1̶7̶ | 6̶ |
| B | 0̶ | 1̶6̶ | 0̶ | 1̶0̶ |
| C | 2̶0̶ | 1̶1̶ | 9̶ | 0̶ |
| D | 9̶ | 0̶ | 1̶4̶ | 1̶ |

Table 13

Step 6

Make assignments in accordance with the rules previously described which result in the following assignments:

$C$ to $Z$

$D$ to $X$

$A$ to $W$

$B$ to $Y$

Step 7

Calculate contribution to be gained from the assignments.

| | |
|---|---|
| $C$ to $Z$ | 30 |
| $D$ to $X$ | 28 |
| $A$ to $W$ | 25 |
| $B$ to $Y$ | 53 |
| | #136 |

Notes

i. It will be apparent that maximizing assignment problems can be solved in virtually the same manner as minimizing problems.
ii. The solution methods given are suitable for any size of matrix. If a problem is as small as the illustration used in this chapter, it can probably be solved merely by inspection.

### 3.3    Dummy Rows and Dummy Columns

The solution procedure to assignment problems requires that the number of rows in the table equal the number of columns. However, when the number of people or objects to be assigned does not equal the number of tasks or clients or machine listed in the columns, we have more rows and columns. We simply add a dummy column. If the number of tasks that need to be done exceeds the number of people available, we add a dummy row. This creates matrix of equal dimensions and allows us to solve the problem as before. Since the dummy tasks or person is really non-existent, it is reasonable to enter zeros in its row or column.

| Person | Projects | | | |
|:---:|:---:|:---:|:---:|:---:|
|  | *I* | *II* | *III* | Dummy ( # ) |
| *A* | 11 | 14 | 6 | 0 |
| *B* | 8 | 10 | 11 | 0 |
| *C* | 9 | 12 | 7 | 0 |
| *D* | 10 | 13 | 8 | 0 |

## 4.0. CONCLUSION

The linear programming models, the complete enumeration method, the Hungarian method as well as the zero-one integer programming method are used to solve assignment problems.

## 5.0. SUMMARY

1.    Assignment problems, like the transportation problems are allocation problem as well as linear programming problems.
2.    Assignment problem is a type of transportation problem.
3.    Assignment problem can be used to minimize cost when executing N jobs from N sources, assigned to N individual at N locations.
4.    If we have unequal rows or column we introduce dummy row or column to balance up the problem.

**6.0. TUTOR MARKED ASSIGNMENT**

1. "The assignment problem is a type of allocation problem". Required:
    a. What do you understand by an assignment problem?
    b. Explain one technique for solving such problems, illustrating your answer by means of a simple example.
2. Three jobs are to be assigned to three machines; each job can be assigned to one and only one machine. The cost in # of job on each machine is given in the following table:

| Machine | Jobs | | |
|---|---|---|---|
| | *I* | *II* | *III* |
| *A* | 20 | 26 | 30 |
| *B* | 10 | 15 | 19 |
| *C* | 17 | 14 | 12 |

You are required to:

a) Determine the job assignments which will minimize cost.
b) Determine the minimum cost.

**7.0. REFERENCES/FURTHER READINGS**

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